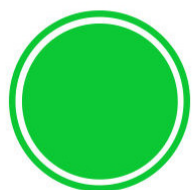


This is a compiled collection of **entire Irodov solution** videos (within JEE scope) available on my youtube channel  [Arihant Senior](#)

Release date:
April-2023

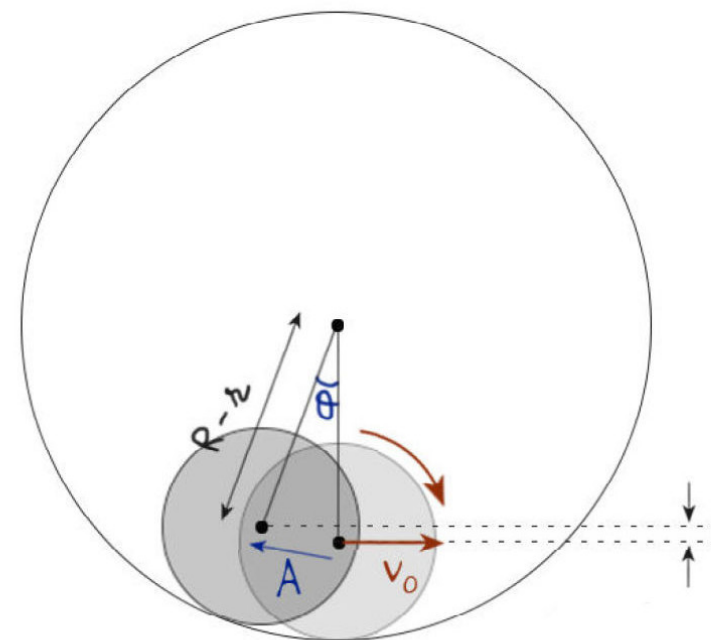
Currently, I am solving another book **Pathfinder** on  [Arihant Senior](#).
Do subscribe to get your daily dose of advanced physics!



Revision Book for **IRODOV XI**

Ankit Singhvi
Dual Degree, IIT Madras

Mechanics
Thermodynamics
Oscillations
Waves



I keep updating the videos and the book. So please download **updated version** in August 2023 from  [Arihant Senior](#)

1.1	1	1.32	18	1.65	35	1.97	52	1.138	69	1.168	86	1.198	103	1.226	120	137	1.292	154		
1.2	1	1.33	18	1.66	35	1.98	52	1.139	69	1.169	86	1.199	103	1.227	120	1.258	137	1.293	154	
1.3	2	1.34	19	1.67	36	1.99	53	1.140	70	1.170	87		104		121	1.259	138	1.294	155	
1.4	2	1.35	19	1.68	36		1.100	53	1.141	70	1.171	87	1.200	104	1.228	121	1.260	138	1.295	155
1.5	3	1.36	20	1.69	37		1.101	54	1.142	71	1.172	88	1.201	105	1.229	122	1.261	139	1.296	156
1.6	3	1.37	20	1.70	37		1.102	54	1.143	71	1.173	88	1.202	105	1.230	122	1.262	139	1.297	156
1.7	4	1.38	21	1.71	38		1.103	55	1.144	72	1.174	89	1.203	106	1.231	123	1.263	140	1.298	157
1.8	4	1.39	21	1.72	38		1.104	55	1.145	72	1.175	89	1.204	106	1.232	123	1.264	140	1.299	157
1.9	5	1.40	22	1.73	39		1.105	56	1.146	73	1.176	90	1.205	107		124		141		158
1.10	5	1.41	22	1.74	39		1.106	56	1.147	73	1.177	90	1.206	107	1.233	124	1.265	141	1.305	158
1.11	6	1.42	23	1.75	40		1.107	57	1.148	74	1.178	91		108	1.234	125		142		159
1.12	6	1.43	23	1.76	40		1.108	57	1.149	74	1.179	91	1.207	108	1.235	125		142	1.306,	159
1.13	7	1.44	24	1.77	41		1.110	58	1.150	75	1.180	92	1.208	109	1.236	126	1.266	143	1.307	
1.14	7	1.45	24	1.78	41		1.118	58	1.151	75	1.181	92	1.209	109	1.237	126	1.267	143	1.308	160
	8	1.46	25	1.79	42		1.119	59	1.152	76	1.182	93	1.210	110	1.238	127	1.269	144		160
1.15	8	1.47	25	1.80	42		1.120	59		76	1.183	93	1.211	110	1.239	127	1.270	144	1.309,	161
1.16	9	1.48	26	1.81	43		1.121	60		77	1.184	94		111	1.240	128	1.271	145	1.310	
1.17	9	1.49	26	1.82	43		1.122	60	1.153	77	1.185	94	1.212	111	1.241	128		145	1.311	161
1.18	10	1.50	27	1.83	44		1.123	61		78	1.186	95		112	1.242	129	1.272	146		162
1.19	10	1.51	27	1.84	44		1.124	61	1.154	78	1.187	95	1.213	112	1.243	129	1.273	146	1.312,	162
1.20	11	1.52	28	1.85	45		1.125	62	1.155	79	1.188	96	1.214	113	1.244	130	1.274	147	1.313	
1.21	11	1.53	28		45		1.126	62	1.156	79		96	1.215	113	1.245	130	1.275	147	1.314	163
	12	1.54	29	1.86	46		1.127	63	1.157	80	1.189	97		114	1.246	131		148	1.315	163
1.22	12	1.55	29	1.87	46		1.128	63	1.158	80	1.190	97	1.216	114	1.247	131	1.276	148	1.316	164
1.23	13	1.56	30	1.88	47		1.129	64	1.159	81	1.191	98		115	1.248	132	1.277	149	1.317	164
1.24	13	1.57	30	1.89	47		1.130	64	1.160	81		98	1.217	115	1.249	132	1.278	149	1.318	165
1.25	14	1.58	31	1.90	48		1.131	65	1.161	82		99	1.218	116	1.250	133	1.279	150	1.319	165
1.26	14	1.59	31	1.91	48			65	1.162	82	1.192	99	1.219	116	1.251	133	1.280	150	1.320	166
1.27	15	1.60	32		49		1.132	66	1.163	83	1.193	100		117	1.252	134		151	1.321	166
	15	1.61	32	1.92	49		1.133	66	1.164	83	1.194	100	1.220	117	1.253	134	1.281	151	1.322	167
1.28	16		33	1.93	50		1.134	67	1.165	84	1.195	101	1.221	118	1.254	135	1.282	152	1.323	167
1.29	16	1.62	33	1.94	50		1.135	67		84	1.196	101	1.222	118	1.255	135		152	1.324	168
1.30	17	1.63	34	1.95	51		1.136	68	1.166	85	1.197	102	1.223	119	1.256	136	1.290	153		168
1.31	17	1.64	34	1.96	51		1.137	68	1.167	85		102	1.224	119	1.257	136	1.291	153	1.326	169

1.327	169	2.26	186	2.66	203	2.126	220	2.170	237	4.8	254	4.39	271	4.73	288	4.110	305	4.154	321
1.328	170	2.27	187	2.67	204	2.127	220		237	4.9	254	4.40	271	4.74	288	4.111	305	4.155	322
1.329	170	2.28	187		204	2.130	221	2.171	238	4.10	255	4.41	272	4.75	289	4.112	306		322
	171	2.29	188	2.69,	205	2.131	221		238	4.11	255	4.42	272	4.76	289	4.113,	306		323
1.330	171	2.30	188	2.70		2.132	222	2.172	239	4.12	256		273	4.77	290	4.114		4.156	323
1.331	172	2.31	189	2.71	205	2.133	222	2.173	239	4.13	256	4.43	273	4.78	290	4.115	307		324
1.332	172	2.32	189	2.72	206	2.134	223	2.174	240	4.14	257	4.44	274	4.79	291	4.120	307	4.157	324
	173	2.33	190	2.73	206	2.135	223	2.175	240	4.15	257	4.45	274	4.80	291	4.121	308	4.158	325
	173	2.34	190	2.74	207	2.136	224	2.176	241	4.16	258		275	4.81	292	4.122	308	4.159	325
1.333	174	2.35	191	2.75	207	2.137	224	2.177	241		258	4.46	275		292		309	4.160	326
	174	2.36	191	2.76	208	2.138	225	2.178	242	4.17	259	4.47	276	4.82	293	4.123	309		326
1.334	175	2.37	192	2.77	208	2.139	225	2.179	242	4.18	259	4.48	276		293	4.124	310		327
	175	2.38	192	2.78	209	2.142	226	2.180	243	4.19	260	4.49	277	4.84	294		310	4.161	327
1.339	176	2.39	193	2.79	209	2.143	226	2.181	243	4.20	260	4.50	277		294	4.125	311	4.162	328
2.1	176	2.40	193	2.83	210	2.144	227	2.182	244	4.21	261	4.51	278	4.89	295	4.126	311	4.170	328
2.2	177	2.41	194	2.85	210	2.145	227	2.183	244	4.22	261	4.52	278	4.94	295	4.127	312	4.171	329
2.3	177	2.43	194	2.87	211	2.146	228	2.184	245	4.23	262	4.53	279	4.95	296	4.128	312	4.172	329
2.4	178	2.44	195	2.92	211	2.147	228	2.246	245	4.24	262	4.54	279		296	4.129	313	4.173	330
2.5	178	2.45	195	2.113	212	2.148	229	2.247	246	4.25	263	4.55	280	4.96	297	4.130	313	4.174	330
2.6	179	2.46	196	2.114	212	2.149	229	2.248	246	4.26	263	4.56	280	4.98	297	4.131	314	4.175	331
2.7	179	2.47	196	2.115	213	2.150	230	2.249	247	4.27	264	4.57	281		298		314	4.176	331
2.8	180	2.48	197	2.116	213	2.151	230	2.250	247	4.28	264	4.58	281		298	4.132	315	4.177	332
2.9	180	2.49	197	2.117	214	2.152	231	2.251	248	4.29	265	4.59	282	4.99	299		315	4.178	332
2.10	181	2.50	198	2.118	214	2.153	231	2.254	248	4.30	265	4.60	282	4.100	299	4.133	316	4.179	333
2.11	181	2.51	198	2.119	215	2.160	232	2.255	249	4.31	266	4.61	283	4.101	300		316	4.180	333
2.12	182	2.52	199	2.120	215	2.161	232	2.256	249	4.32	266	4.62	283	4.102	300	4.140	317	4.183	334
2.13	182	2.53	199		216	2.162	233	2.257	250	4.33	267	4.63	284		301	4.141	317	4.185	334
2.14	183	2.54	200	2.121	216	2.163	233	4.1	250	4.34	267	4.64	284	4.103	301	4.142	318		335
2.15	183	2.55	200		217	2.164	234	4.2	251	4.35	268	4.65	285	4.104	302	4.143	318	4.186	335
2.16	184	2.56	201	2.122	217	2.165	234	4.3	251		268	4.66	285	4.105	302	4.144	319	4.187	336
2.17	184	2.62	201		218	2.166	235	4.4	252	4.36	269	4.67	286	4.106	303	4.146	319		
2.18	185	2.63	202	2.123	218	2.167	235	4.5	252	4.37	269	4.68	286	4.107	303	4.147	320		
2.19	185	2.64	202	2.124	219	2.168	236	4.6	253	4.38	270	4.70	287	4.108	304	4.150	320		
2.20	186	2.65	203	2.125	219	2.169	236	4.7	253		270	4.72	287	4.109	304	4.153	321		

1.1. A motorboat going downstream overcame a raft at a point A; $\tau = 60$ min later it turned back and after some time passed the raft at a distance $l = 6.0$ km from the point A. Find the flow velocity assuming the duty of the engine to be constant.

• Raft is floating on river: $v_{\text{river}} = v_{\text{raft}}$

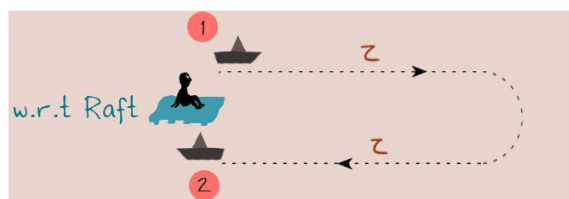
Analysing w.r.t raft/river: which means raft/river are at rest.



Boat travels to and fro with same velocity, just like a car on a stationary road



$$t_{\text{forward}} = t_{\text{backward}} = \tau$$



Now time varies equally in all reference frames.

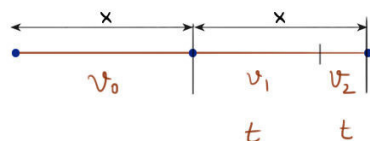
• Time Between events 1 and 2 = Time taken by raft to move distance l .

$$\Rightarrow 2\tau = \frac{l}{v_{\text{raft}}}$$

$$\Rightarrow v_{\text{raft}} = \frac{l}{2\tau} = v_{\text{river}}$$

Youtube / Arihant Senior

1.2. A point traversed half the distance with a velocity v_0 . The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.



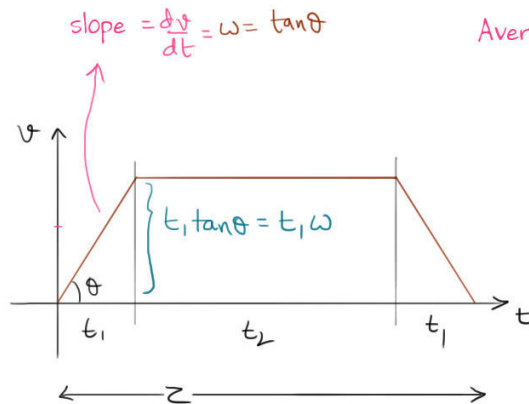
$$\langle v \rangle = \frac{\text{Total disp}}{\text{Total time}} = \frac{2x}{\frac{x}{v_0} + 2t}$$

$$v_1 t + v_2 t = x$$

$$= \frac{2x}{\frac{x}{v_0} + 2 \frac{x}{v_1 + v_2}}$$

$$= \frac{2v_0(v_1 + v_2)}{v_1 + v_2 + 2v_0}$$

1.3. A car starts moving rectilinearly, first with acceleration $w = 5.0 \text{ m/s}^2$ (the initial velocity is equal to zero), then uniformly, and finally, decelerating at the same rate w , comes to a stop. The total time of motion equals $\tau = 25 \text{ s}$. The average velocity during that time is equal to $\langle v \rangle = 72 \text{ km per hour}$. How long does the car move uniformly? $t_2 = ?$



Average velocity: $\bar{v} = \frac{\text{Area}}{\text{time}}$

$\Rightarrow \bar{v} = \frac{\frac{1}{2}(\tau + t_2)(wt_1)}{\tau}$

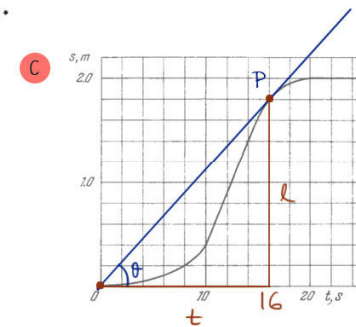
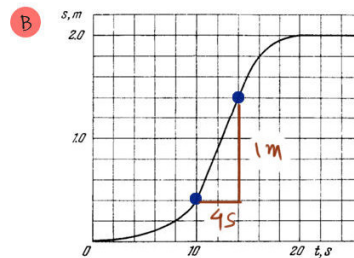
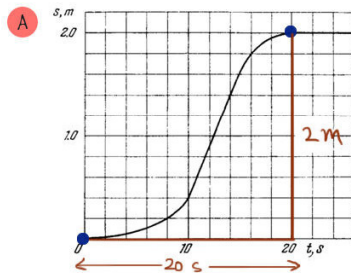
$= \frac{1}{2} \frac{(\tau + t_2)}{\tau} w \frac{(\tau - t_2)}{2}$

$\Rightarrow 4\tau \bar{v} = \tau^2 - t_2^2 \Rightarrow t_2 = \sqrt{\tau^2 - \frac{4\tau \bar{v}}{w}}$

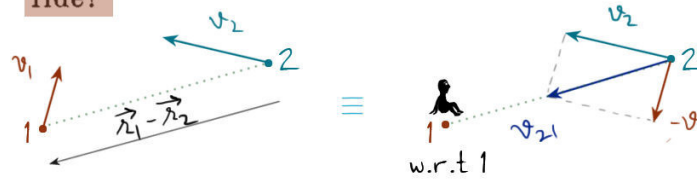
Youtube / Arihant Senior

1.4. A point moves rectilinearly in one direction. Fig. 1.1 shows the distance s traversed by the point as a function of the time t . Using the plot find:

- the average velocity of the point during the time of motion;
- the maximum velocity;
- the time moment t_0 at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.



1.5. Two particles, 1 and 2, move with constant velocities $\vec{v}_1$ and $\vec{v}_2$. At the initial moment their radius vectors are equal to $\vec{r}_1$ and $\vec{r}_2$. How must these four vectors be interrelated for the particles to collide?



In order to collide:

2 must be heading towards 1, w.r.t 1, as relative acceleration is zero.

$$\text{i.e. } \hat{v}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$\Rightarrow \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

Relative acceleration complications:

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$\vec{s}_{21} = \vec{u}_{21}t + \frac{1}{2}\vec{a}_{21}t^2$$

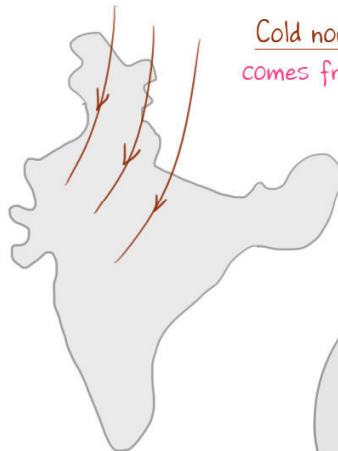
$$\vec{r}_1 - \vec{r}_2 = \vec{u}_2 - \vec{u}_1 \quad \vec{a}_2 - \vec{a}_1$$

$$\text{As, } \vec{r}_{21} = \vec{r}_2 - \vec{r}_1$$

$$\vec{s}_{21} = \vec{r}_1 - \vec{r}_2$$

Youtube / Arihant Senior

1.6. A ship moves along the equator to the east with velocity $v_0 = 30$ km/hour. The southeastern wind blows at an angle $\phi = 60^\circ$ to the equator with velocity $v = 15$ km/hour. Find the wind velocity v' relative to the ship and the angle ϕ' between the equator and the wind direction in the reference frame fixed to the ship.



Cold north wind comes from north $\Rightarrow$ Direction of wind is south $\vec{v}_w \downarrow$

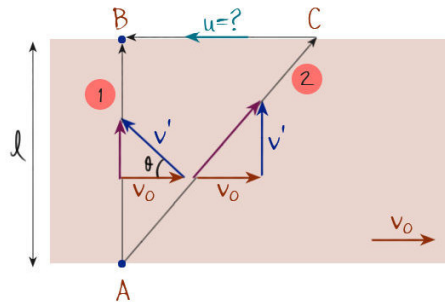
South eastern wind comes from SE $\Rightarrow$ Direction of wind is North western $\vec{v}_w \nwarrow$

$$|\vec{v}_{ws}| = |\vec{v}_w - \vec{v}_s| = \sqrt{v_w^2 + v_s^2 + 2v_w v_s \cos \phi}$$

$$\tan \phi' = \frac{v_w \sin \phi}{v_s + v_w \cos \phi} = \frac{15 \sin 60}{30 + 15 \cos 60}$$

Youtube / Arihant Senior

1.7. Two swimmers leave point A on one bank of the river to reach point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to point B . What was the velocity u of his walking if both swimmers reached the destination simultaneously? The stream velocity $v_0 = 2.0$ km/hour and the velocity v' of each swimmer with respect to water equals 2.5 km per hour.



let the time taken by either path be t_0

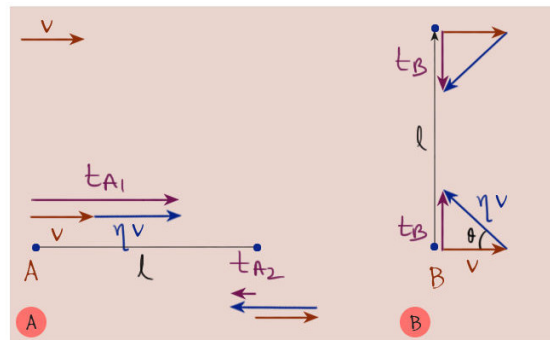
Path 1: $t_0 = \frac{l}{v' \sin \theta}$ $\sin \theta = \sqrt{1 - \cos^2 \theta}$
 $= \sqrt{1 - \left(\frac{v_0}{v'}\right)^2}$

Path 2: $t_0 = t_{\text{water}} + t_{\text{land}}$
 $= \frac{l}{v'} + \frac{\text{drift}}{u} \rightarrow \text{drift} = v_0 \left(\frac{l}{v'}\right)$

$$t_0 = t_0 \Rightarrow \frac{l}{v' \sqrt{v'^2 - v_0^2}} = \frac{l}{v'} + \frac{1}{u} \left(\frac{v_0 l}{v'} \right) \Rightarrow u = \frac{v_0 \sqrt{v'^2 - v_0^2}}{v' - \sqrt{v'^2 - v_0^2}}$$

Youtube / Arihant Senior

1.8. Two boats, A and B , move away from a buoy anchored at the middle of a river along the mutually perpendicular straight lines: the boat A along the river, and the boat B across the river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats τ_A / τ_B if the velocity of each boat with respect to water is $\eta = 1.2$ times greater than the stream velocity.



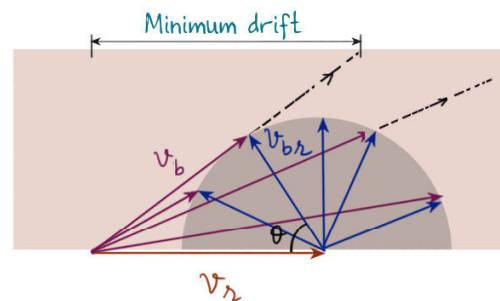
A $\tau_A = t_{A1} + t_{A2}$
 $= \frac{l}{v + \eta v} + \frac{l}{\eta v - v} = \frac{2\eta l}{v(\eta^2 - 1)}$

B $\tau_B = 2t_B = \frac{2l}{\eta v \sin \theta}$ $\sin \theta = \sqrt{1 - \cos^2 \theta}$
 $= \sqrt{1 - \frac{1}{\eta^2}}$

$$\Rightarrow \frac{\tau_A}{\tau_B} = \frac{2\eta l}{v(\eta^2 - 1)} \cdot \frac{\eta v \sqrt{\eta^2 - 1}}{2l} = \frac{\eta}{\sqrt{\eta^2 - 1}}$$

Youtube / Arihant Senior

1.9. A boat moves relative to water with a velocity which is $n = 2.0$ times less than the river flow velocity. At what angle to the stream direction must the boat move to minimize drifting?



Its given that: $v_{b||r} = \frac{1}{n} v_r$

From diagram, for minimum drift:

$$\cos \theta = \frac{v_{b||r}}{v_b} = \frac{1}{n}$$

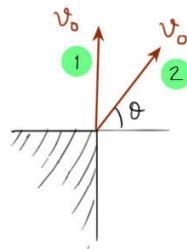
Angle made by boat wrt stream: $= \pi - \theta$



$$= \pi - \cos^{-1} \frac{1}{n} = \sin^{-1} \frac{1}{n} + \frac{\pi}{2}$$

Youtube / Arihant Senior

1.10. Two bodies were thrown simultaneously from the same point: one, straight up, and the other, at an angle of $\theta = 60^\circ$ to the horizontal. The initial velocity of each body is equal to $v_0 = 25$ m/s. Neglecting the air drag, find the distance between the bodies $t = 1.70$ s later.

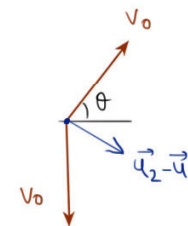


$$\vec{S}_{21} = \vec{u}_{21} t + \frac{1}{2} \vec{a}_{21} t^2$$

$$(\vec{r}_2 - \vec{r}_1) = (\vec{u}_2 - \vec{u}_1) t = [v_0 \cos \theta \hat{i} + (v_0 - v_0 \sin \theta)(-\hat{j})] t$$

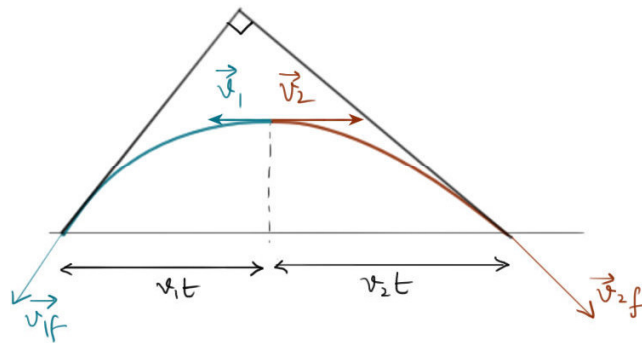
$$|\vec{r}_1 - \vec{r}_2| = v_0 \sqrt{\cos^2 \theta + 1 + \sin^2 \theta - 2 \sin \theta} t$$

$$= v_0 \sqrt{2(1 - \sin \theta)} t$$



Youtube / Arihant Senior

1.11. Two particles move in a uniform gravitational field with an acceleration g . At the initial moment the particles were located at one point and moved with velocities $v_1 = 3.0 \text{ m/s}$ and $v_2 = 4.0 \text{ m/s}$ horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.



$$\vec{v}_{1f} = \vec{v}_1 + \vec{g}t$$

$$\vec{v}_{2f} = \vec{v}_2 + \vec{g}t$$

Given, $v_{1f} \perp v_{2f} \Rightarrow \vec{v}_{1f} \cdot \vec{v}_{2f} = 0$

$$\Rightarrow (\vec{v}_1 + \vec{g}t) \cdot (\vec{v}_2 + \vec{g}t) = 0$$

$$\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{g}t + \vec{v}_2 \cdot \vec{g}t + g^2 t^2 = 0$$

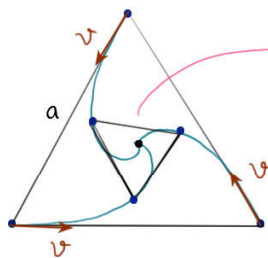
$$\downarrow$$

$$-v_1 v_2 + g^2 t^2 = 0 \Rightarrow t = \frac{\sqrt{v_1 v_2}}{g}$$

∴ Distance between them: $v_1 t + v_2 t = (v_1 + v_2) \frac{\sqrt{v_1 v_2}}{g}$

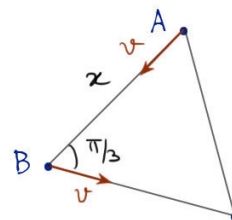
Youtube / Arihant Senior

1.12. Three points are located at the vertices of an equilateral triangle whose side equals a . They all start moving simultaneously with velocity v constant in modulus, with the first point heading continually for the second, the second for the third, and the third for the first. How soon will the points converge?



By symmetry three points always form the vertices of an equilateral triangle of decreasing side length, and eventually meet at original centroid.

Considering any arbitrary equilateral triangle during the motion, velocity of approach between two points A and B:



$$-\frac{dx}{dt} = v + v \cos \pi/3$$

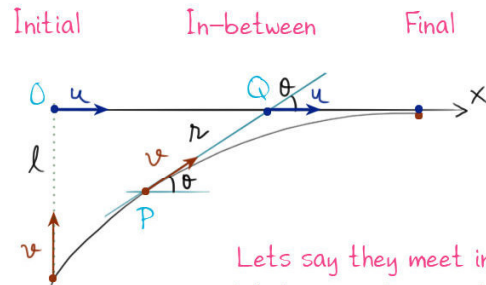
$$-dx = v \left(1 + \frac{1}{2}\right) dt$$

$$\int_a^0 dx = \frac{3v}{2} \int_0^t dt$$

$$\Rightarrow t = \frac{2a}{3v}$$

Youtube / Arihant Senior

1.13. Point A moves uniformly with velocity v so that the vector v is continually "aimed" at point B which in its turn moves rectilinearly and uniformly with velocity $u < v$. At the initial moment of time $v \perp u$ and the points are separated by a distance l . How soon will the points converge?



Lets say they meet in time t_0
Writing equations in two directions:

A Displacement in X direction:

$$\int_0^{t_0} v \cos \theta dt = \int_0^{t_0} u dt$$

$$\Rightarrow v \int_0^{t_0} \cos \theta dt = u t_0 \quad (1)$$

B Displacement along line joining both points:

$$\frac{dr}{dt} = u \cos \theta - v$$

$$dr = u \cos \theta dt - v dt$$

$$\int_l^0 dr = \int_0^{t_0} u \cos \theta dt - \int_0^{t_0} v dt$$

$$\Rightarrow l = v t_0 - u \int_0^{t_0} \cos \theta dt \quad (2)$$

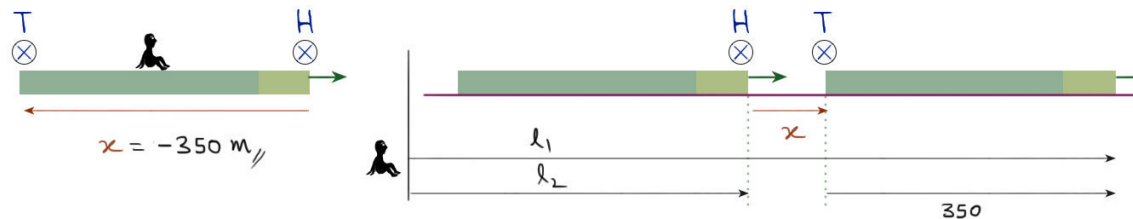
$$1, 2 \Rightarrow t_0 = \frac{vl}{v^2 - u^2}$$

Youtube / Arihant Senior

1.14. A train of length $l = 350$ m starts moving rectilinearly with constant acceleration $w = 3.0 \cdot 10^{-2}$ m/s²; $t = 30$ s after the start the locomotive headlight is switched on (event 1), and $\tau = 60$ s after that event the tail signal light is switched on (event 2). Find the distance between these events in the reference frames fixed to the train and to the Earth. How and at what constant velocity V relative to the Earth must a certain reference frame K move for the two events to occur in it at the same point?

A w.r.t train

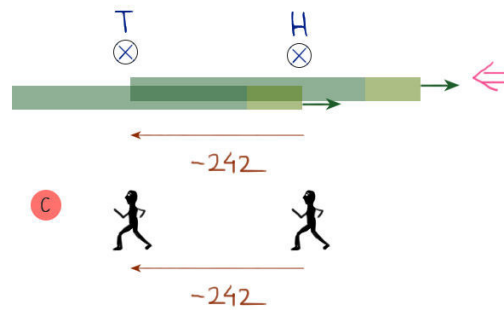
B w.r.t ground



$$x = l_1 - l_2 - 350$$

$$= \frac{1}{2} (0.03) 90^2 - \frac{1}{2} (0.03) 30^2 - 350 = -242 \text{ m}$$

Youtube / Arihant Senior



-ve sign means overlap!
i.e the train has not even traveled its own length yet.

For both events occurring at the same place to observer.....

Observer's change in position = event's change in position.

$$\therefore \vec{x}_{\text{obsv}} = -242$$

$$\vec{v}_0 t = -242$$

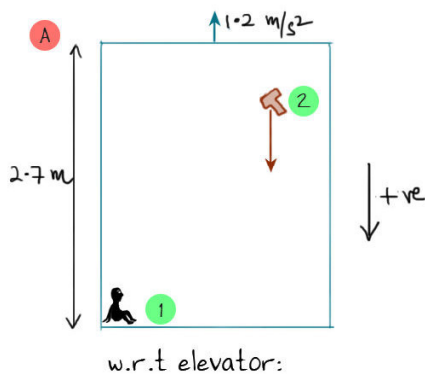
$$\vec{v}_0 = \frac{-242}{60} = -4 \text{ m/s} //$$

Youtube / Arihant Senior

1.15. An elevator car whose floor-to-ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 m/s^2 ; 2.0 s after the start a bolt begins falling from the ceiling of the car. Find:

(a) the bolt's free fall time;

(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.



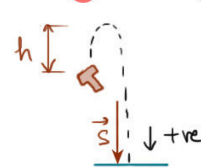
w.r.t elevator:

$$\vec{s}_{21} = \vec{u}_{21} + \frac{1}{2} \vec{a}_{21} t^2$$

$$+2.7 = 0 + \frac{1}{2} (9.8 - (-1.2)) t^2 \Rightarrow t = \sqrt{\frac{2 \times 2.7}{11}} = 0.7 \text{ s} //$$

Ground frame

B w.r.t ground



Displacement:

Distance:

$$d = 2h + s$$

$$= \frac{2u^2}{2g} + s$$

$$= \frac{2(2.4)^2}{2(9.8)} + 0.72 = 1.3 \text{ m} //$$

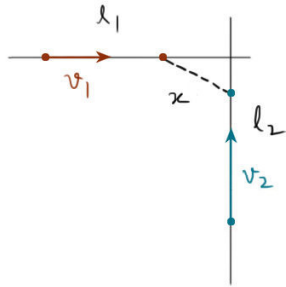
$$\vec{u} = \vec{a}t = (-1.2)(2) = -2.4$$

$$\vec{s} = \vec{u}t + \frac{1}{2} \vec{a}t^2$$

$$\Rightarrow \vec{s} = 0.72 \text{ m} //$$

Youtube / Arihant Senior

1.16. Two particles, 1 and 2, move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines toward the intersection point O . At the moment $t = 0$ the particles were located at the distances l_1 and l_2 from the point O . How soon will the distance between the particles become the smallest? What is it equal to?



At general time t :

$$x = \sqrt{(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2}$$

For minimum x :

$$\frac{dx}{dt} = 0 \Rightarrow 2(l_1 - v_1 t)(-v_1) + 2(l_2 - v_2 t)(-v_2) = 0$$

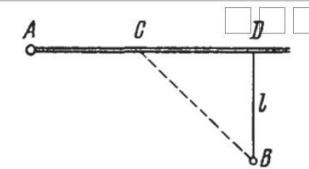
$$\Rightarrow t = \frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$$

Putting t

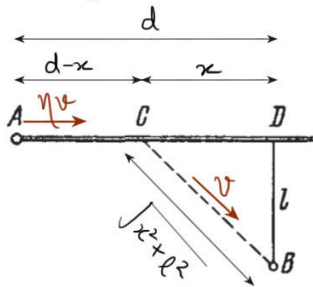
$$\Rightarrow x_{\min} = \frac{|l_1 v_2 - l_2 v_1|}{\sqrt{v_1^2 + v_2^2}}$$

Youtube / Arihant Senior

1.17. From point A located on a highway (Fig. 1.2) one has to get by car as soon as possible to point B located in the field at a distance l from the highway. It is known that the car moves in the field η times slower than on the highway. At what distance from point D one must turn off the highway?



Let length AD be d and it turns at distance x from D



Time taken for the entire journey:

$$t = \frac{d-x}{\eta v} + \frac{\sqrt{x^2 + l^2}}{v}$$

For minimum time to reach B :

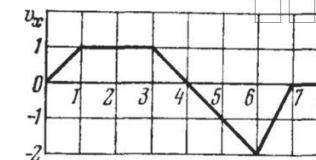
$$\frac{dt}{dx} = 0, \Rightarrow -\frac{1}{\eta v} + \frac{1}{v} \frac{x}{\sqrt{x^2 + l^2}} = 0$$

$$\eta^2 x^2 = x^2 + l^2$$

$$x = \frac{l}{\sqrt{\eta^2 - 1}}$$

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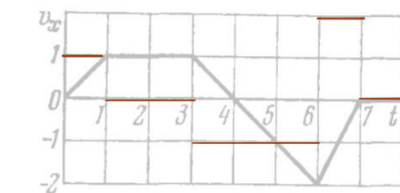
1.18. A point travels along the x axis with a velocity whose projection v_x is presented as a function of time by the plot in Fig. 1.3. Assuming the coordinate of the point $x = 0$ at the moment $t = 0$, draw the approximate time dependence plots for the acceleration w_x , the x coordinate, and the distance covered s .



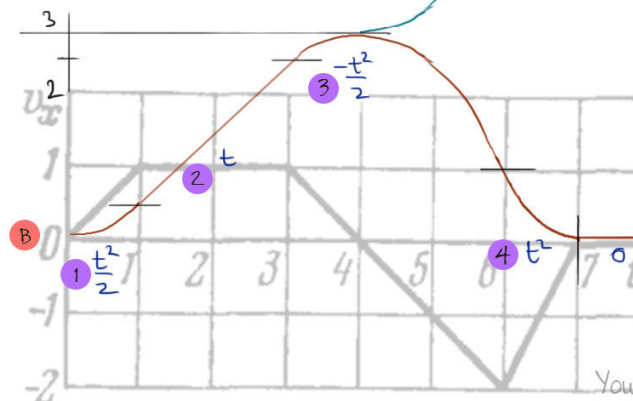
Acceleration: Slope of $v-t$

Displacement: Area under $v-t$

Distance: Magnitude of Area under $v-t$



A acceleration



displacement

$$= \int v dt$$

$$\textcircled{1} \int t dt = \frac{t^2}{2}$$

$$\textcircled{2} \int dt = t$$

$$\textcircled{3} \int -t dt = -\frac{t^2}{2}$$

$$\textcircled{4} \int 2t dt = t^2$$

Youtube / Arihant Senior

1.19. A point traversed half a circle of radius $R = 160$ cm during time interval $\tau = 10.0$ s. Calculate the following quantities averaged over that time:

(a) the mean velocity $\langle v \rangle$; $\rightarrow$ average speed

(b) the modulus of the mean velocity vector $|\langle \vec{v} \rangle|$; $\rightarrow$ | average velocity |

(c) the modulus of the mean vector of the total acceleration $|\langle \vec{w} \rangle|$ $\rightarrow$ | average acceleration | if the point moved with constant tangential acceleration.

A $\langle v \rangle = \frac{\text{distance}}{\text{time}} = \frac{\pi R}{\tau}$

B $|\langle \vec{v} \rangle| = \frac{|\text{displacement}|}{\text{time}} = \frac{2R}{\tau}$

C $|\langle \vec{w} \rangle| = \frac{|\vec{v}_f - \vec{v}_i|}{\text{time}}, \quad \text{Let tangential acceleration be } a_t$

$$\pi R = \frac{1}{2} a_t \tau^2 \Rightarrow a_t = \frac{2\pi R}{\tau^2}$$

$$\Rightarrow \left. \begin{array}{l} v_f = a_t \tau = \frac{2\pi R}{\tau} \\ v_i = 0 \end{array} \right\} |\langle \vec{w} \rangle| = \frac{2\pi R}{\tau^2}$$

Youtube / Arihant Senior

1.20. A radius vector of a particle varies with time t as $\vec{r} = \vec{a}t(1 - \alpha t)$, where $\vec{a}$ is a constant vector and α is a positive factor.

Find:

- (a) the velocity $\vec{v}$ and the acceleration $\vec{w}$ of the particle as functions of time;
 (b) the time interval Δt taken by the particle to return to the initial points, and the distance s covered during that time.

A $\vec{r} = \vec{a}t(1 - \alpha t)$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{a}(1 - 2\alpha t)$$

$$\vec{w} = \frac{d\vec{v}}{dt} = -2\alpha\vec{a}$$

B Its motion in one dimension as position, velocity and acceleration are all parallel to each other. So we can write the equations as:

$$r = at(1 - \alpha t)$$

$$v = a(1 - 2\alpha t)$$

Particle is at origin then $r = 0$

$$\text{i.e. } at(1 - \alpha t) = 0 \Rightarrow t = 0, 1/\alpha \Rightarrow \Delta t = \frac{1}{\alpha}$$

$$\text{At forward most point, } v = 0, \Rightarrow a(1 - 2\alpha t) = 0, t = \frac{1}{2\alpha}$$

Distance covered = $2 \times$ Forward distance

$$= 2 \times r_{t=\frac{1}{2\alpha}} = 2 \times \left[a \frac{1}{2\alpha} \left(1 - \alpha \frac{1}{2\alpha} \right) \right] = \frac{a}{2\alpha}$$

1.21. At the moment $t = 0$ a particle leaves the origin and moves in the positive direction of the x axis. Its velocity varies with time as $\vec{v} = \vec{v}_0(1 - t/\tau)$, where $\vec{v}_0$ is the initial velocity vector whose modulus equals $v_0 = 10.0$ cm/s; $\tau = 5.0$ s. Find:

- (a) the x coordinate of the particle at the moments of time 6.0, 10, and 20 s;
 (b) the moments of time when the particle is at the distance 10.0 cm from the origin;
 (c) the distance s covered by the particle during the first 4.0 and 8.0 s; draw the approximate plot $s(t)$.

$$v = \frac{dx}{dt} = v_0 \left(1 - \frac{t}{\tau} \right)$$

$$\Rightarrow \int_0^x dx = \int_0^t v_0 \left(1 - \frac{t}{\tau} \right) dt$$

$$\Rightarrow x = v_0 \left(t - \frac{t^2}{2\tau} \right)$$

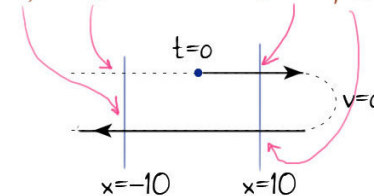
A $x = 10 \left(t - \frac{t^2}{10} \right)$

$t=6 \Rightarrow x = 0.24$
 $t=10 \Rightarrow x = 0$ (Returns to origin)
 $t=20 \Rightarrow x = -4$

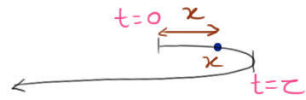
B $\pm 10 = 10 \left(t - \frac{t^2}{10} \right)$

$$t^2 - 10t - 10 = 0, t^2 - 10t + 10 = 0$$

$$t = 11, -0.9 \quad t = 1.1, 9$$

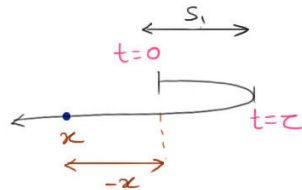


C $v = v_0 \left(1 - \frac{t}{2\tau}\right)$ $x = v_0 \left(t - \frac{t^2}{2\tau}\right)$



$t < 2\tau$

Distance: $S = v_0 \left(t - \frac{t^2}{2\tau}\right)$



$t > 2\tau$

Distance: $S = 2S_1 + (-x)$

$= 2 \left(v_0 \frac{2\tau}{2} \right) - \left[v_0 \left(t - \frac{t^2}{2\tau} \right) \right]$

$= v_0 2\tau - v_0 t \left(1 - \frac{t}{2\tau} \right)$

$S_1 = v_0 \left(2\tau - \frac{2\tau^2}{2\tau} \right) = \frac{v_0 2\tau}{2}$

Youtube / Arihant Senior

1.22. The velocity of a particle moving in the positive direction of the x axis varies as $v = \alpha \sqrt{x}$, where α is a positive constant. Assuming that at the moment $t = 0$ the particle was located at the point $x = 0$, find:

(a) the time dependence of the velocity and the acceleration of the particle;

(b) the mean velocity of the particle averaged over the time that the particle takes to cover the first s metres of the path.

$v = \frac{dx}{dt} = \alpha \sqrt{x}$

$\Rightarrow \int_0^x \frac{dx}{\sqrt{x}} = \int_0^t \alpha dt$

$\Rightarrow 2\sqrt{x} = \alpha t$

A $x = \frac{\alpha^2 t^2}{4}$ differentiating w.r.t time $\rightarrow v = \frac{dx}{dt} = \frac{\alpha^2 t}{2}$ differentiating w.r.t time $\rightarrow a = \frac{dv}{dt} = \frac{\alpha^2}{2}$

B $v_{av} = \frac{\text{total distance}}{\text{time}}, S = \frac{\alpha^2 t^2}{4}$ (Given)

$v_{av} = \frac{S}{t} = \frac{S}{2\sqrt{S}/\alpha} = \frac{\alpha \sqrt{S}}{2}$

Youtube / Arihant Senior

1.23. A point moves rectilinearly with deceleration whose modulus depends on the velocity v of the particle as $w = a\sqrt{v}$, where a is a positive constant. At the initial moment the velocity of the point is equal to v_0 . What distance will it traverse before it stops? What time will it take to cover that distance?

A $a_{cc} = v \frac{dv}{dx} = -a\sqrt{v}$
 Indicates deceleration

$$\Rightarrow \int_{v_0}^v \sqrt{v} dv = -a \int_0^x dx$$

$$\Rightarrow \frac{2}{3} (v^{3/2} - v_0^{3/2}) = -ax$$

$$\Rightarrow \text{At } v=0, x = \frac{2}{3a} v_0^{3/2}$$

B $\frac{dv}{dt} = -a\sqrt{v}$
 $\int_{v_0}^v \frac{dv}{\sqrt{v}} = -\int_0^t a dt$

$$2(\sqrt{v} - \sqrt{v_0}) = -at$$

$$\text{At } v=0, t = \frac{2\sqrt{v_0}}{a}$$

Youtube / Arihant Senior

1.24. A radius vector of a point A relative to the origin varies with time t as $\mathbf{r} = at\mathbf{i} - bt^2\mathbf{j}$, where a and b are positive constants, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find:

- the equation of the point's trajectory $y(x)$; plot this function;
- the time dependence of the velocity $\mathbf{v}$ and acceleration $\mathbf{w}$ vectors, as well as of the moduli of these quantities;
- the time dependence of the angle α between the vectors $\mathbf{w}$ and $\mathbf{v}$;
- the mean velocity vector averaged over the first t seconds of motion, and the modulus of this vector.

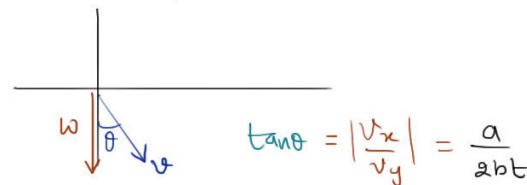
$$\mathbf{r} = at\mathbf{i} - bt^2\mathbf{j} \xrightarrow[\text{w.r.t } t]{\text{differentiate}} \mathbf{v} = a\mathbf{i} - 2bt\mathbf{j} \xrightarrow{\text{diff}} \mathbf{w} = -2b\mathbf{j}$$

$$x = at \quad y = -bt^2$$

Eliminating t

$$y = -b\left(\frac{x^2}{a^2}\right)$$

(equation of trajectory)



$$\vec{v}_{av} = \frac{\vec{r}_f - \vec{r}_i}{t} = \frac{at\mathbf{i} - bt^2\mathbf{j}}{t} = a\mathbf{i} - bt\mathbf{j}$$

Youtube / Arihant Senior

1.25. A point moves in the plane xy according to the law $x = at$, $y = at(1 - \alpha t)$, where a and α are positive constants, and t is time. Find:

- the equation of the point's trajectory $y(x)$; plot this function;
- the velocity v and the acceleration w of the point as functions of time;
- the moment t_0 at which the velocity vector forms an angle $\pi/4$ with the acceleration vector.

$x = at$
 $y = at(1 - \alpha t)$

differentiate w.r.t t

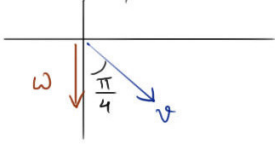
$v_x = a$
 $v_y = a(1 - 2\alpha t)$
 $\vec{v} = a\hat{i} + a(1 - 2\alpha t)\hat{j}$

diff

$w_x = 0$
 $w_y = -2a\alpha$
 $\vec{w} = -2a\alpha\hat{j}$

Eliminating t

$y = \frac{x}{a}(1 - \frac{\alpha x}{a})$
 $y = x(1 - \frac{\alpha x}{a})$
 (equation of trajectory)



$\frac{\pi}{4} \Rightarrow |v_x| = |v_y| \Rightarrow a = a(1 - 2\alpha t)$
 $t = 0, \frac{1}{2\alpha}$ // Youtube / Arihant Senior

1.26. A point moves in the plane xy according to the law $x = a \sin \omega t$, $y = a(1 - \cos \omega t)$, where a and ω are positive constants. Find:

- the distance s traversed by the point during the time τ ;
- the angle between the point's velocity and acceleration vectors.

$x = a \sin \omega t$
 $y = a(1 - \cos \omega t)$

differentiate

$v_x = a\omega \cos \omega t$
 $v_y = a\omega \sin \omega t$
 $\vec{v} = a\omega(\cos \omega t \hat{i} + \sin \omega t \hat{j})$

differentiate

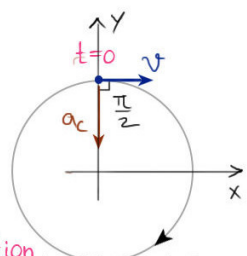
$a_x = -a\omega^2 \sin \omega t$
 $a_y = a\omega^2 \cos \omega t$
 $\vec{a} = -a\omega^2(\sin \omega t \hat{i} - \cos \omega t \hat{j})$

$v_x^2 + v_y^2 = a^2 \omega^2$
 $|\vec{v}| = a\omega$ (A constant)

distance $s = vt = a\omega t$

Angle between velocity and acceleration:

$\cos \theta = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}| |\vec{a}|} = \frac{-a^2 \omega^3 \cos \omega t \sin \omega t + a^2 \omega^3 \sin \omega t \cos \omega t}{a\omega \cdot a\omega^2} = 0 \Rightarrow \theta = \pi/2$



Uniform circular motion // Youtube / Arihant Senior

1.27. A particle moves in the plane xy with constant acceleration w directed along the negative y axis. The equation of motion of the particle has the form $y = ax - bx^2$, where a and b are positive constants. Find the velocity of the particle at the origin of coordinates.

$$\dot{x} = \frac{dx}{dt} \quad \ddot{x} = \frac{d^2x}{dt^2}$$

$$\dot{y} = \frac{dy}{dt} \quad \ddot{y} = \frac{d^2y}{dt^2}$$

Given, $y = ax - bx^2$, $\ddot{y} = -w$

diff ↓

$$\dot{y} = a\dot{x} - 2bx\dot{x} \quad (1)$$

diff ↓

$$\ddot{y} = a\ddot{x} - 2b(x\ddot{x} + \dot{x}^2)$$

$$\Rightarrow -w = a\ddot{x} - 2b(x\ddot{x} + \dot{x}^2)$$

As no acceleration in X direction

$$\Rightarrow -w = -2b\dot{x}^2 \Rightarrow \dot{x} = \sqrt{\frac{w}{2b}} \quad \text{(constant)}$$

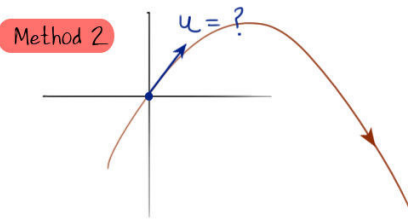
Put in 1 at $x=0 \rightarrow \dot{y} = a\dot{x} = a\sqrt{\frac{w}{2b}} \quad (2)$

1 & 2 $\Rightarrow$

$$v|_{\text{origin}} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$= \sqrt{\frac{w}{2b}(1+a^2)}$$

Youtube / Arihant Senior



We know,
Equation of trajectory for projectile path is

$$y = x \tan \theta - \frac{wx^2}{2u^2 \cos^2 \theta}$$

Comparing it with equations 1 and 2,

$$a = \tan \theta, \quad b = \frac{w}{2u^2 \cos^2 \theta}$$

Eliminating θ $\sec^2 \theta = 1 + \tan^2 \theta$

$$b = \frac{w}{2u^2} (1 + a^2)$$

$$\Rightarrow u = \sqrt{\frac{w(1+a^2)}{2b}}$$

Given,

$$\frac{d^2y}{dt^2} = -w \quad (1)$$

$$y = ax - bx^2 \quad (2)$$

Youtube / Arihant Senior

1.28. A small body is thrown at an angle to the horizontal with the initial velocity $\vec{v}_0$. Neglecting the air drag, find:

- (a) the displacement of the body as a function of time $\vec{r}(t)$;
 (b) the mean velocity vector $\langle \vec{v} \rangle$ averaged over the first t seconds and over the total time of motion.

A We know that,

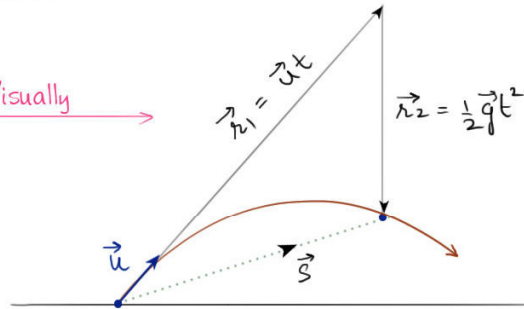
$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2 \quad \xrightarrow{\text{Visually}}$$

Displacement

$$\Rightarrow \vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$$

B $\langle \vec{v} \rangle = \left| \frac{\vec{r}}{t} \right|$

$$= \vec{v}_0 + \frac{1}{2}\vec{g}t \quad (1)$$



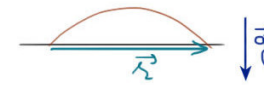
C 1 & 2 $\Rightarrow$

$$\begin{aligned} \langle \vec{v} \rangle \text{ for full flight} &= \vec{v}_0 + \frac{1}{2}\vec{g} \left(\frac{2}{g}(\vec{v}_0 \cdot \vec{g}) \right) \\ &= \vec{v}_0 - (\vec{v}_0 \cdot \vec{g}) \frac{\vec{g}}{g^2} \end{aligned}$$

C When it falls down to ground: $\vec{r} \perp \vec{g}$

$$\Rightarrow (\vec{v}_0 t + \frac{1}{2}\vec{g}t^2) \cdot \vec{g} = 0$$

$$(\vec{v}_0 \cdot \vec{g})t + \frac{1}{2}gt^2 = 0 \Rightarrow t = \frac{-2(\vec{v}_0 \cdot \vec{g})}{g^2} \quad (2)$$



Youtube / Arihant Senior

1.29. A body is thrown from the surface of the Earth at an angle α to the horizontal with the initial velocity v_0 . Assuming the air drag to be negligible, find:

- (a) the time of motion;
 (b) the maximum height of ascent and the horizontal range; at what value of the angle α they will be equal to each other;
 (c) the equation of trajectory $y(x)$, where y and x are displacements of the body along the vertical and the horizontal respectively;
 (d) the curvature radii of trajectory at its initial point and at its peak.

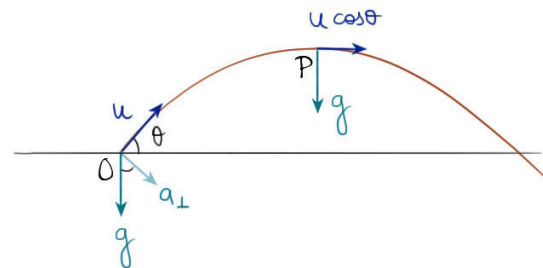
D We know,

$$\text{Radius of curvature} = R_c = \frac{v^2}{a_{\perp}}$$

$\therefore$

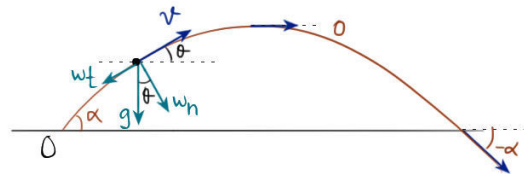
$$R_c|_0 = \frac{u^2}{g \cos \theta}$$

$$R_c|_P = \frac{u^2 \cos^3 \theta}{g}$$

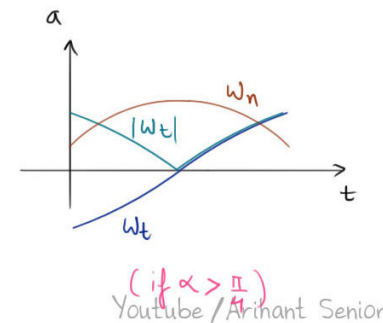
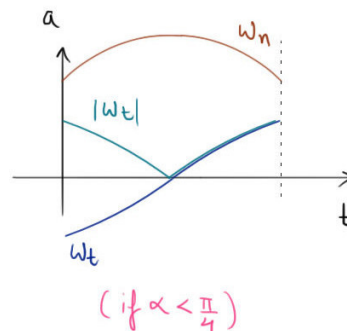


Youtube / Arihant Senior

1.30. Using the conditions of the foregoing problem, draw the approximate time dependence of moduli of the normal w_n and tangent w_t acceleration vectors, as well as of the projection of the total acceleration vector w_v on the velocity vector direction.

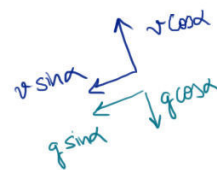
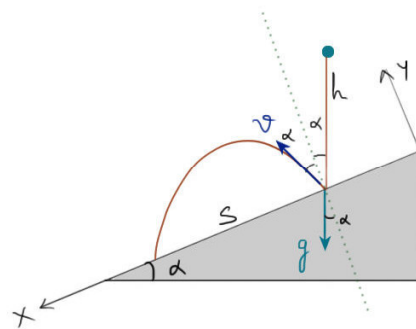


$$\begin{aligned} w_n &= g \cos \theta \\ w_t &= -g \sin \theta \end{aligned} \quad \theta: \alpha \text{ to } 0 \text{ to } -\alpha$$



Youtube/Arihant Senior

1.31. A ball starts falling with zero initial velocity on a smooth inclined plane forming an angle α with the horizontal. Having fallen the distance h , the ball rebounds elastically off the inclined plane. At what distance from the impact point will the ball rebound for the second time?



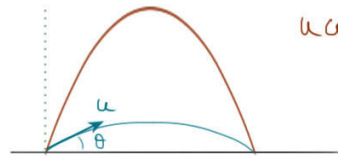
$$v = \sqrt{2gh}$$

$$t = \frac{2v \cos \alpha}{g} = \frac{2(v \cos \alpha)}{g \cos \alpha} = \frac{2v}{g}$$

$$\begin{aligned} S &= u_x t + \frac{1}{2} g_x t^2 \\ &= v \sin \alpha \cdot \frac{2v}{g} + \frac{1}{2} g \sin \alpha \left(\frac{2v}{g} \right)^2 \\ &= \frac{4v^2 \sin \alpha}{g} \\ &= 4(2gh) \frac{\sin \alpha}{g} = 8h \sin \alpha \end{aligned}$$

Youtube/Arihant Senior

1.32. A cannon and a target are 5.10 km apart and located at the same level. How soon will the shell launched with the initial velocity 240 m/s reach the target in the absence of air drag?



$$u \cos \theta \cdot t = R \quad (1)$$

$$t = \frac{2u \sin \theta}{g} \quad (2)$$

Eliminating θ

$$\frac{R^2}{u^2 t^2} + \frac{g^2 t^2}{4u^2} = 1$$

$$\Rightarrow g^2 t^4 - 4u^2 t^2 + 4R^2 = 0$$

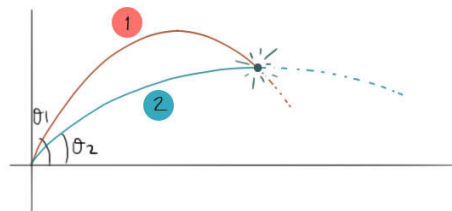
$$R = 5.1 \text{ km}$$

$$u = 240 \text{ m/s}$$

$$\Rightarrow t = \sqrt{\frac{4u^2 \pm \sqrt{16u^4 - 4g^2 4R^2}}{2g^2}}$$

Youtube / Arihant Senior

1.33. A cannon fires successively two shells with velocity $v_0 = 250 \text{ m/s}$; the first at the angle $\theta_1 = 60^\circ$ and the second at the angle $\theta_2 = 45^\circ$ to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells.



$$\text{Trajectory for 1: } y = x \tan \theta_1 - \frac{gx^2}{2v_0^2 \cos^2 \theta_1} \quad (1)$$

$$\text{Trajectory for 2: } y = x \tan \theta_2 - \frac{gx^2}{2v_0^2 \cos^2 \theta_2} \quad (2)$$

$$\text{Time for 1: } t = \frac{x}{v_0 \cos \theta_1} \quad (3)$$

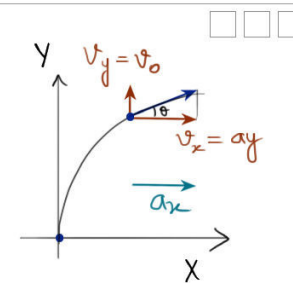
$$\text{Time for 2: } t - \Delta t = \frac{x}{v_0 \cos \theta_2} \quad (4)$$

$$\text{Four equations, four variables: } \Rightarrow \Delta t = \frac{2v_0}{g} \left[\frac{\sin(\theta_1 - \theta_2)}{\cos \theta_1 + \cos \theta_2} \right]$$

Youtube / Arihant Senior

1.34. A balloon starts rising from the surface of the Earth. The ascension rate is constant and equal to v_0 . Due to the wind the balloon gathers the horizontal velocity component $v_x = ay$, where a is a constant and y is the height of ascent. Find how the following quantities depend on the height of ascent:

- (a) the horizontal drift of the balloon $x(y)$;
 (b) the total, tangential, and normal accelerations of the balloon.



A Given,

$$v_x = ay \quad v_y = v_0$$

$$\Rightarrow \frac{dx}{dt} = ay \quad \Rightarrow \frac{dy}{dt} = v_0$$

$$\Rightarrow dx = ay dt$$

$$\Rightarrow dx = ay \frac{dy}{v_0}$$

$$\Rightarrow \int_0^x dx = \frac{a}{v_0} \int_0^y y dy \Rightarrow x = \frac{ay^2}{2v_0}$$

B $a_{total} = a_x = \frac{dv_x}{dt} = a \frac{dy}{dt} = av_0$

$$a_t = a_x \cos \theta = \frac{a_x \cdot ay}{\sqrt{a^2 y^2 + v_0^2}}$$

$$a_n = a_x \sin \theta = \frac{a_x \cdot v_0}{\sqrt{a^2 y^2 + v_0^2}}$$

Youtube / Arihant Senior

1.35. A particle moves in the plane xy with velocity $\mathbf{v} = a\mathbf{i} + bx\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes, and a and b are constants. At the initial moment of time the particle was located at the point $x = y = 0$. Find:

- (a) the equation of the particle's trajectory $y(x)$;
 (b) the curvature radius of trajectory as a function of x .

A $v_x = a \quad | \quad v_y = bx$

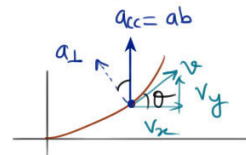
$$\frac{dx}{dt} = a \quad \frac{dy}{dt} = bx$$

$$\frac{dx}{dy} = \frac{a}{bx}$$

$$\Rightarrow \int_0^x bx dx = \int_0^y a dy$$

$$\Rightarrow \frac{bx^2}{2a} = y$$

Method 1 B $a = \frac{dv}{dt} = b \frac{dx}{dt} \hat{j} = ab \hat{j}$



Radius of trajectory:

$$R_c = \frac{v^2}{a_{\perp}} = \frac{v^2}{ab \cos \theta} = \frac{v^3}{ab v_x} = \frac{(a^2 + b^2 x^2)^{3/2}}{ab \cdot a} = \frac{(a^2 + b^2 x^2)^{3/2}}{a^2 b}$$

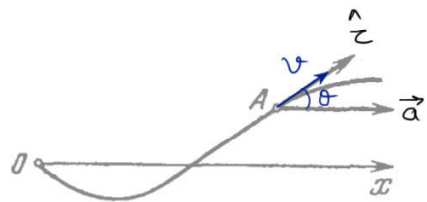
Method 2 B

Using formula for ROC:

$$R_c = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{bx}{a}\right)^2\right]^{3/2}}{\frac{b}{a}}$$

Youtube / Arihant Senior

1.36. A particle A moves in one direction along a given trajectory with a tangential acceleration $w_t = a\tau$, where a is a constant vector coinciding in direction with the x axis (Fig. 1.4), and τ is a unit vector coinciding in direction with the velocity vector at a given point. Find how the velocity of the particle depends on x provided that its velocity is negligible at the point $x = 0$.



We know that,

Tangential acceleration: $w_t = v \frac{dv}{ds}$

Speed
Distance

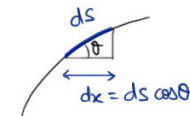
Sol: Given: $w_t = \vec{a} \cdot \hat{\tau} = a \cos \theta$

$$\Rightarrow v \frac{dv}{ds} = a \cos \theta$$

$$\Rightarrow v dv = a ds \cos \theta$$

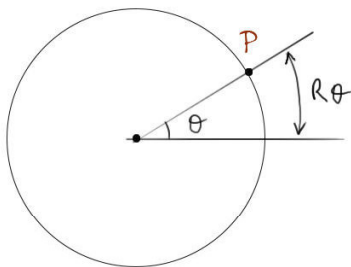
$$\Rightarrow v dv = a dx$$

$$\Rightarrow \int_0^v v dv = \int_0^x a dx \Rightarrow \frac{v^2}{2} = ax \Rightarrow v = \sqrt{2ax}$$



Youtube / Arihant Senior

1.37. A point moves along a circle with a velocity $v = at$, where $a = 0.50 \text{ m/s}^2$. Find the total acceleration of the point at the moment when it covered the n -th ($n = 0.10$) fraction of the circle after the beginning of motion.



$$v = at$$

diff. ↓

$$a_t = a \text{ (constant) } ①$$

$$\therefore R\theta = \frac{1}{2}at^2 \quad \left[\theta = n(2\pi) \right]$$

$$\Rightarrow R(2n\pi) = \frac{1}{2}at^2$$

$$\Rightarrow t = \sqrt{\frac{4n\pi R}{a}}$$

∴ At point P,

$$v = at = \sqrt{a4n\pi R} \quad ②$$

$$\vec{a}_{\text{total}} = \vec{a}_n + \vec{a}_t$$

$$= \sqrt{\left(\frac{v^2}{R}\right)^2 + a^2}$$

$$\Rightarrow a_{\text{total}} = a \sqrt{(4n\pi)^2 + 1}$$

Youtube / Arihant Senior

1.38. A point moves with deceleration along the circle of radius R so that at any moment of time its tangential and normal accelerations are equal in moduli. At the initial moment $t = 0$ the velocity of the point equals v_0 . Find:

- (a) the velocity of the point as a function of time and as a function of the distance covered s ;
 (b) the total acceleration of the point as a function of velocity and the distance covered.

A

$$a_t = a_n$$

$$\Rightarrow -\frac{dv}{dt} = \frac{v^2}{R}$$

$$\Rightarrow \int_{v_0}^v \frac{dv}{v^2} = -\frac{1}{R} \int_0^t dt$$

$$\Rightarrow \frac{1}{v} - \frac{1}{v_0} = \frac{t}{R}$$

$$\Rightarrow v = \frac{v_0 R}{v_0 t + R} //$$

$$a_t = a_n$$

$$\Rightarrow -v \frac{dv}{ds} = \frac{v^2}{R}$$

$$\Rightarrow -\int_{v_0}^v \frac{dv}{v} = \int_0^s \frac{ds}{R}$$

$$\Rightarrow \ln \frac{v}{v_0} = -\frac{s}{R}$$

$$\Rightarrow v = v_0 e^{-s/R} //$$

B Total $a = \sqrt{a_t^2 + a_n^2}$

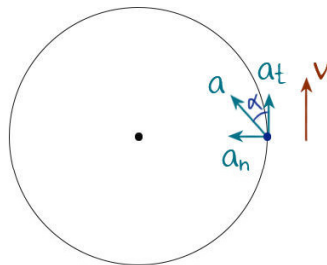
$$= \sqrt{2} a_n \quad (\because a_n = a_t)$$

$$= \sqrt{2} \left(\frac{v^2}{R} \right) //$$

$$= \sqrt{2} \frac{v_0^2}{R^2} e^{-2s/R} //$$

Youtube / Arihant Senior

1.39. A point moves along an arc of a circle of radius R . Its velocity depends on the distance covered s as $v = a\sqrt{s}$, where a is a constant. Find the angle α between the vector of the total acceleration and the vector of velocity as a function of s .



$$\angle \text{ Between } a \text{ \& } v = \alpha = \tan^{-1} \frac{a_n}{a_t}$$

$$a_n = \frac{v^2}{R} = \frac{a^2 s}{R} \quad \Bigg| \quad a_t = v \frac{dv}{ds} = a\sqrt{s} \cdot \frac{a}{2\sqrt{s}} = \frac{a^2}{2}$$

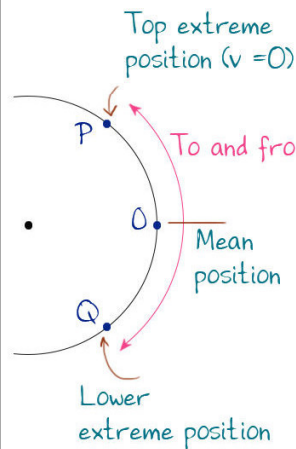
$$\Rightarrow \alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{\frac{a^2 s}{R}}{\frac{a^2}{2}} = \tan^{-1} \frac{2s}{R} //$$

Youtube / Arihant Senior

1.40. A particle moves along an arc of a circle of radius R according to the law $l = a \sin \omega t$, where l is the displacement from the initial position measured along the arc, and a and ω are constants. Assuming $R = 1.00$ m, $a = 0.80$ m, and $\omega = 2.00$ rad/s, find:

(a) the magnitude of the total acceleration of the particle at the points $l = 0$ and $l = \pm a$;

(b) the minimum value of the total acceleration w_{min} and the corresponding displacement l_m .



A $l = a \sin \omega t$

$$v = \frac{dl}{dt} = a\omega \cos \omega t$$

$$a_t = \frac{dv}{dt} = -a\omega^2 \sin \omega t$$

Total acceleration: $\sqrt{a_n^2 + a_t^2}$

$$\sqrt{\left(\frac{v^2}{R}\right)^2 + a_t^2}$$

Point O $\frac{a^2 \omega^2}{R} \quad 0 \Rightarrow a_0 = \frac{a^2 \omega^2}{R}$

P, Q $0 \quad a\omega^2 \Rightarrow a_p = a\omega^2$

B $a_{net} = \sqrt{a_n^2 + a_t^2} = \sqrt{\left(\frac{v^2}{R}\right)^2 + a_t^2}$

$$= \sqrt{\left(\frac{a^2 \omega^2 \cos^2 \omega t}{R}\right)^2 + a^2 \omega^4 \sin^2 \omega t}$$

$$= a\omega^2 \sqrt{\frac{a^2}{R^2} \cos^2 \omega t + \sin^2 \omega t}$$

Minimum at: $\cos \omega t = \frac{R}{\sqrt{2}a}$

$$\Rightarrow a_{min} = a\omega^2 \sqrt{1 - \left(\frac{R}{2a}\right)^2}$$

at $l = a \sin \omega t = a \sqrt{1 - \left(\frac{R}{\sqrt{2}a}\right)^2}$

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1.41. A point moves in the plane so that its tangential acceleration $w_t = a$, and its normal acceleration $w_n = bt^4$, where a and b are positive constants, and t is time. At the moment $t = 0$ the point was at rest. Find how the curvature radius R of the point's trajectory and the total acceleration w depend on the distance covered s .

$$a_t = \frac{dv}{dt} = a \text{ (constant)}$$

$$a_n = \frac{v^2}{R_c} = bt^4 \quad 3$$

$$\Rightarrow v = at \quad 1$$

$$\& s = \frac{1}{2}at^2 \quad 2$$

A Eliminating v and t from 1, 2, 3..

$$R_c = \frac{a^2 t^2}{bt^4} = \frac{a^2}{bt^2} = \frac{a^2}{b \cdot 2s/a} = \frac{a^3}{2bs}$$

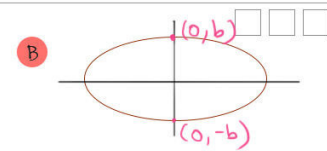
B $a_{net} = \sqrt{a_t^2 + a_n^2} = \sqrt{a^2 + b^2 t^8} \xrightarrow{t \text{ from } 2} \sqrt{a^2 + b^2 \left(\frac{2s}{a}\right)^4}$

Youtube / Arihant Senior

1.42. A particle moves along the plane trajectory $y(x)$ with velocity v whose modulus is constant. Find the acceleration of the particle at the point $x = 0$ and the curvature radius of the trajectory at that point if the trajectory has the form

(a) of a parabola $y = ax^2$;

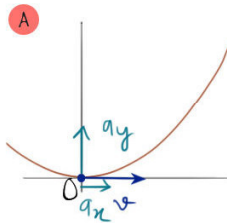
(b) of an ellipse $(x/a)^2 + (y/b)^2 = 1$; a and b are constants here.



$a_t = a_n = 0$

∴ speed is constant: $a_t = 0$

$y = ax^2$



$\Rightarrow v_y = \frac{dy}{dt} = 2ax \frac{dx}{dt}$

$\Rightarrow a_y = \frac{d^2y}{dt^2} = 2a \left(\frac{dx}{dt} \right)^2 + 2ax \frac{d^2x}{dt^2}$
 $= 2av^2$

At origin,

$a_t = a_n = 0$

∴ $a_{net} = \sqrt{a_x^2 + a_y^2} = 2av^2$

$R_c = \frac{v^2}{a_{\perp}} = \frac{v^2}{a_y} = \frac{v^2}{2av^2} = \frac{1}{2a}$

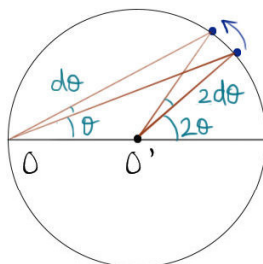
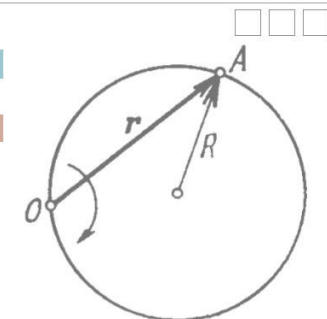
diff $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 $\frac{2x}{a^2} \frac{dx}{dt} + \frac{2y}{b^2} \frac{dy}{dt} = 0$
 $\frac{1}{a^2} \left(\frac{dx}{dt} \right)^2 + \frac{1}{b^2} \left(\frac{dy}{dt} \right)^2 = 0$
 $\Rightarrow \frac{v^2}{a^2} \pm \frac{b}{b^2} a_y = 0 \Rightarrow a_y = \pm \frac{bv^2}{a^2}$

∴ $a_{net} = \sqrt{a_x^2 + a_y^2} = \pm \frac{bv^2}{a^2}$

$R_c = \frac{v^2}{a_{\perp}} = \frac{v^2}{a_y} = \frac{v^2 a^2}{\pm bv^2} = \pm \frac{a^2}{b}$

Youtube / Arihant Senior

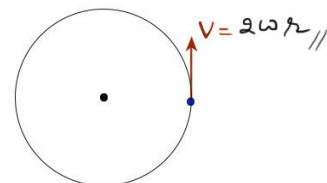
1.43. A particle A moves along a circle of radius $R = 50$ cm so that its radius vector r relative to the point O (Fig. 1.5) rotates with the constant angular velocity $\omega = 0.40$ rad/s. Find the modulus of the velocity of the particle, and the modulus and direction of its total acceleration.



If angular velocity is ω relative to O , i.e. $\frac{d\theta}{dt}$

Then its angular velocity is 2ω relative to O' i.e. $\frac{d(2\theta)}{dt}$

Uniform circular motion with angular velocity 2ω .



$a_{total} = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R$

Youtube / Arihant Senior

1.44. A wheel rotates around a stationary axis so that the rotation angle ϕ varies with time as $\phi = at^2$, where $a = 0.20 \text{ rad/s}^2$. Find the total acceleration w of the point A at the rim at the moment $t = 2.5 \text{ s}$ if the linear velocity of the point A at this moment $v = 0.65 \text{ m/s}$.

$$\begin{aligned}\phi &= at^2 \\ \omega &= \frac{d\phi}{dt} = 2at \Rightarrow v = \omega r = 2art \\ \alpha &= \frac{d^2\phi}{dt^2} = 2a \Rightarrow a_t = \alpha r = 2ar \\ a_{\text{net}} &= \sqrt{a_t^2 + a_n^2} \quad a_n = \frac{v^2}{r} \\ &= \sqrt{4a^2 r^2 + \frac{v^4}{r^2}} \\ &= \sqrt{\frac{4a^2 v^2}{4a^2 t^2} + \frac{v^4 4a^2 t^2}{v^2}} \\ &= v \sqrt{\frac{1}{t^2} + 4a^2 t^2} //\end{aligned}$$

Youtube / Arihant Senior

1.45. A shell acquires the initial velocity $v = 320 \text{ m/s}$, having made $n = 2.0$ turns inside the barrel whose length is equal to $l = 2.0 \text{ m}$. Assuming that the shell moves inside the barrel with a uniform acceleration, find the angular velocity of its axial rotation at the moment when the shell escapes the barrel.



Given: l, n, v

To find: $\omega^2 = 2\alpha\theta$

$\Rightarrow \omega^2 = \frac{2v^2}{2l \frac{l}{2\pi n}} \quad (n \cdot 2\pi) = \left(\frac{2\pi n v}{l}\right)^2$

$\Rightarrow \omega = \frac{2\pi n v}{l} //$

$\frac{v^2}{2l} \quad (\because \text{rolling}) \quad \frac{a}{r} \quad (\because n \text{ turns}) \quad l = (2\pi r)n$

$(\because v^2 = u^2 + 2al)$

Youtube / Arihant Senior

1.46. A solid body rotates about a stationary axis according to the law $\theta = at - bt^3$, where $a = 6.0 \text{ rad/s}$ and $b = 2.0 \text{ rad/s}^3$. Find:

(a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop;

(b) the angular acceleration at the moment when the body stops.

$$\theta = at - bt^3$$

$$\omega = \frac{d\theta}{dt} = a - 3bt^2 \Rightarrow \text{stops at } t_s = \sqrt{\frac{a}{3b}}$$

$$\alpha = \frac{d\omega}{dt} = -6bt$$

-ve implies deceleration
(as expected!)

$$\text{A } \langle \omega_{av} \rangle = \frac{\Delta\theta}{\Delta t} = \frac{\theta_s - 0}{t_s - 0}$$

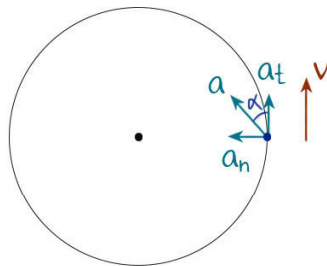
$$= \frac{at_s - bt_s^3}{t_s} = a - bt_s^2 = a - b\left(\frac{a}{3b}\right) = \frac{2a}{3}$$

$$\begin{aligned} \langle \alpha_{av} \rangle &= \frac{\Delta\omega}{\Delta t} = \frac{\omega_s - \omega_0}{t_s} = \frac{-\omega_0}{t_s} \\ &= \frac{-a}{\sqrt{\frac{a}{3b}}} = -\sqrt{3ab} \end{aligned}$$

$$\text{B } \alpha_s = \alpha|_{t_s} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{3ab}$$

Youtube / Arihant Senior

1.47. A solid body starts rotating about a stationary axis with an angular acceleration $\beta = at$, where $a = 2.0 \cdot 10^{-2} \text{ rad/s}^3$. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\alpha = 60^\circ$ with its velocity vector? (Similar to 1.39)



$$\angle \text{Between } a \text{ \& } v = \alpha = \tan^{-1} \frac{a_n}{a_t}$$

$$\begin{aligned} \beta &= at \\ \Rightarrow a_t &= \beta r = a r t \end{aligned}$$

$$\begin{aligned} \beta &= at \\ \Rightarrow \frac{d\omega}{dt} &= at \\ \Rightarrow \int_0^\omega d\omega &= a \int_0^t t dt \\ \Rightarrow \omega &= \frac{at^2}{2} \end{aligned}$$

$$\begin{aligned} \therefore \tan \alpha &= \frac{a_n}{a_t} = \frac{\omega^2 r}{a r t} = \frac{\frac{a^2 t^4}{4}}{a r t} \\ \Rightarrow t &= \left(\frac{4 \tan \alpha}{a} \right)^{1/3} \end{aligned}$$

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1.48. A solid body rotates with deceleration about a stationary axis with an angular deceleration $\beta \propto \sqrt{\omega}$, where ω is its angular velocity. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to ω_0 .

$\beta \propto \sqrt{\omega}$
 $\langle \omega_{av} \rangle = \frac{\Delta \theta}{\Delta t} = \frac{\theta_s}{t_s}$
 Let, $\beta = -k\sqrt{\omega}$ (-ve sign, as decelerating)

$$\frac{d\omega}{dt} = -k\sqrt{\omega}$$

$$\Rightarrow \int_{\omega_0}^0 \frac{d\omega}{\sqrt{\omega}} = -k \int_0^{t_s} dt$$

$$\Rightarrow 2\sqrt{\omega_0} = kt_s \quad (1)$$

$$\omega \frac{d\omega}{d\theta} = -k\sqrt{\omega}$$

$$\Rightarrow \int_{\omega_0}^0 \sqrt{\omega} d\omega = -k \int_0^{\theta_s} d\theta$$

$$\Rightarrow \frac{2}{3} \omega_0^{3/2} = k\theta_s \quad (2)$$

$$\therefore \langle \omega_{av} \rangle = \frac{\theta_s}{t_s} = \frac{(2)}{(1)} = \frac{\frac{2}{3} \omega_0^{3/2}}{2\sqrt{\omega_0}} = \frac{\omega_0}{3} //$$

Youtube / Arihant Senior

1.49. A solid body rotates about a stationary axis so that its angular velocity depends on the rotation angle ϕ as $\omega = \omega_0 - a\theta$, where ω_0 and a are positive constants. At the moment $t = 0$ the angle $\theta = 0$. Find the time dependence of

- the rotation angle;
- the angular velocity.

$$\omega = \frac{d\theta}{dt} = \omega_0 - a\theta$$

$$\Rightarrow \int_0^{\theta} \frac{d\theta}{\omega_0 - a\theta} = \int_0^t dt$$

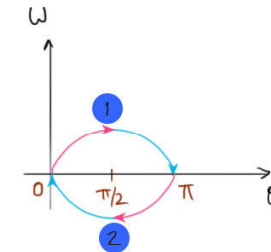
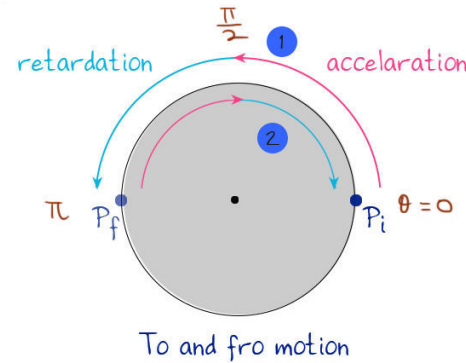
$$\Rightarrow -\frac{1}{a} \ln(\omega_0 - a\theta) \Big|_0^{\theta} = t$$

$$\Rightarrow \frac{\omega_0 - a\theta}{\omega_0} = e^{-at} \Rightarrow \theta = \frac{\omega_0}{a} (1 - e^{-at}) \xrightarrow{\text{differentiate}} \omega = \omega_0 e^{-at} //$$

Youtube / Arihant Senior

1.50. A solid body starts rotating about a stationary axis with an angular acceleration $\beta = \beta_0 \cos \theta$, where β_0 is a constant vector and θ is an angle of rotation from the initial position. Find the angular velocity of the body as a function of the angle θ . Draw the plot of this dependence.

$$\begin{aligned}\beta &= \omega \frac{d\omega}{d\theta} = \beta_0 \cos \theta \\ \Rightarrow \int_0^\omega \omega d\omega &= \beta_0 \int_0^\theta \cos \theta d\theta \\ \Rightarrow \omega^2 &= 2\beta_0 \sin \theta \\ \Rightarrow \omega &= \pm \sqrt{2\beta_0 \sin \theta}\end{aligned}$$

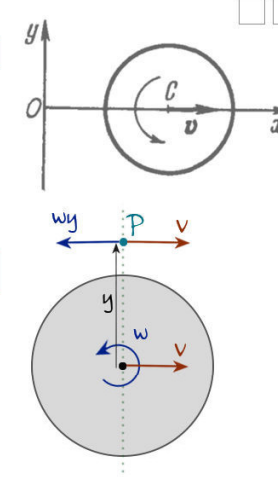


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1.51. A rotating disc (Fig. 1.6) moves in the positive direction of the x axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation, if at the initial moment the axis C of the disc was located at the point O after which it moved

(a) with a constant velocity v , while the disc started rotating counterclockwise with a constant angular acceleration β (the initial angular velocity is equal to zero);

(b) with a constant acceleration a (and the zero initial velocity), while the disc rotates counterclockwise with a constant angular velocity ω .



IOR is a point about which whole body is rotating at a given moment.

It may not be a fixed point.

It is at rest w.r.t ground (it need not be on body).

Its located on a line perpendicular to the velocity at any point on body.

A velocity = v , (constant) Angular accⁿ = β

B accⁿ = a , Angular velocity = ω (constant)

Let IOR be at P

$$\begin{aligned}v_P = 0 &\Rightarrow v = \omega y \Rightarrow v = \beta \frac{x}{v} \cdot y \Rightarrow y = \frac{v^2}{\beta x} \\ &\quad \downarrow \beta t \\ &\quad \frac{x}{v}\end{aligned}$$

Hyperbola

Let IOR be at P

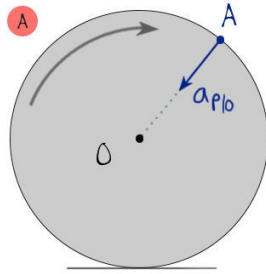
$$\begin{aligned}v_P = 0 &\Rightarrow v = \omega y \Rightarrow \sqrt{2ax} = \omega y \Rightarrow y = \frac{\sqrt{2ax}}{\omega} \\ &\quad \downarrow \\ &\quad [v^2 = 2ax]\end{aligned}$$

Parabola

Youtube / Arihant Senior

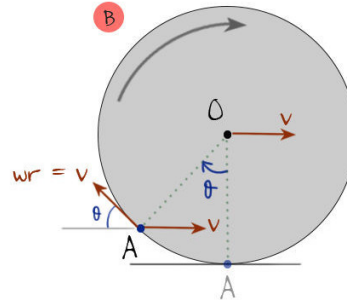
1.52. A point A is located on the rim of a wheel of radius $R = 0.50$ m which rolls without slipping along a horizontal surface with velocity $v = 1.00$ m/s. Find:

- (a) the modulus and the direction of the acceleration vector of the point A ;
 (b) the total distance s traversed by the point A between the two successive moments at which it touches the surface.



$$a_p = a_{p/o} + a_o$$

$$\Rightarrow a_p = \frac{v^2}{R}$$



When A turns by θ

$$|\vec{v}_A| = |\vec{v}_{A/o} + \vec{v}_o| = \sqrt{v^2 + v^2 + 2v \cdot v \cos(\pi - \theta)}$$

$$= v \sqrt{2(1 + \cos \theta)} = 2v \sin \frac{\theta}{2} = 2v \sin \frac{\omega t}{2}$$

Small distance dS traveled by A :

$$dS = v_A dt$$

$$\int_0^S dS = \int_0^{2\pi/\omega} 2v \sin \frac{\omega t}{2} dt$$

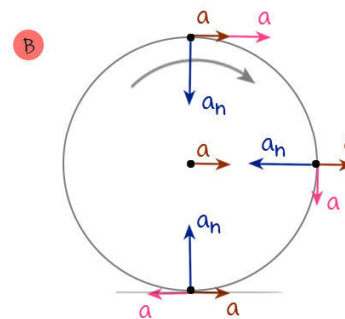
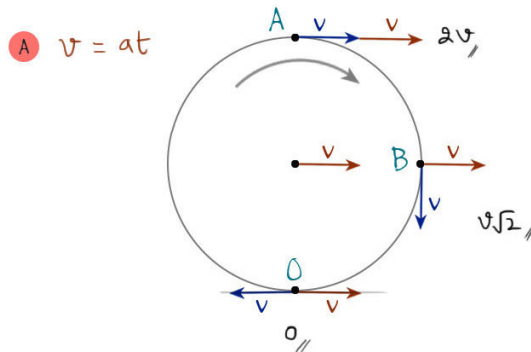
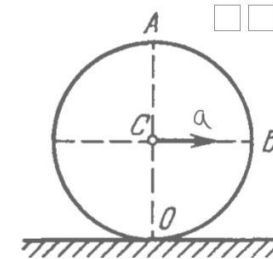
$$\Rightarrow S = \frac{2v}{\omega/2} (-\cos \frac{\omega t}{2}) \Big|_0^{2\pi/\omega}$$

$$= \frac{4v}{\omega/2} (2) = 8R //$$

Youtube / Arihant Senior

1.53. A ball of radius $R = 10.0$ cm rolls without slipping down an inclined plane so that its centre moves with constant acceleration $a = 2.50$ cm/s²; $t = 2.00$ s after the beginning of motion its position corresponds to that shown in Fig. 1.7. Find:

- (a) the velocities of the points A , B , and O ;
 (b) the accelerations of these points.



$$a_n = \frac{v^2}{R} = \frac{a^2 t^2}{R}$$

$$a_t = a \quad (\because \text{rolling})$$

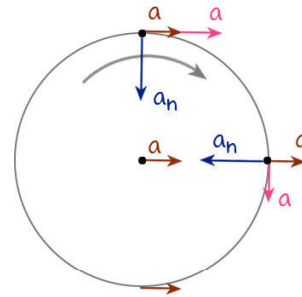
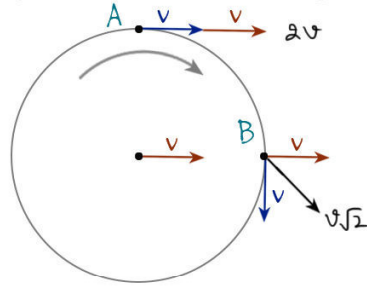
$$A: a = \sqrt{(2a)^2 + \left(\frac{a^2 t^2}{R}\right)^2}$$

$$B: a = \sqrt{\left(\frac{a^2 t^2}{R} - a\right)^2 + a^2}$$

$$C: a = a_n = \frac{a^2 t^2}{R}$$

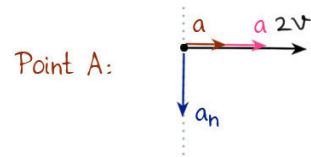
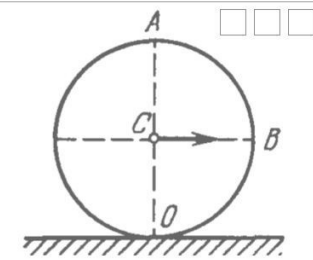
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1.54. A cylinder rolls without slipping over a horizontal plane. The radius of the cylinder is equal to r . Find the curvature radii of trajectories traced out by the points A and B



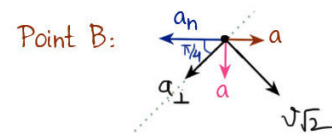
$$a_n = \frac{v^2}{R}$$

$$a_t = a \quad (\because \text{rolling})$$



$$R_c = \frac{\text{velocity}^2}{a_{\perp}} = \frac{(2v)^2}{a_n} = \frac{4v^2}{(v^2/R)} = 4R //$$

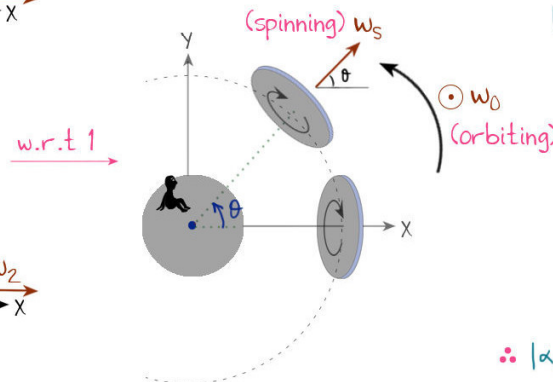
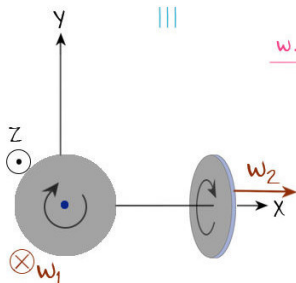
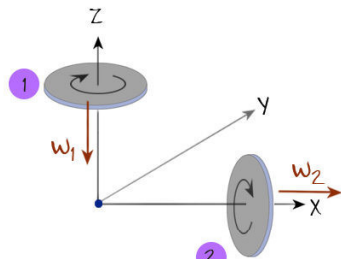
They are independent of acceleration a !



$$R_c = \frac{\text{velocity}^2}{a_{\perp}} = \frac{(v\sqrt{2})^2}{a_n \cos \frac{\pi}{4}} = \frac{v^2 \cdot 2}{\frac{v^2}{R} \cdot \frac{1}{\sqrt{2}}} = 2\sqrt{2}R //$$

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1.55. Two solid bodies rotate about stationary mutually perpendicular intersecting axes with constant angular velocities $\omega_1 = 3.0 \text{ rad/s}$ and $\omega_2 = 4.0 \text{ rad/s}$. Find the angular velocity and angular acceleration of one body relative to the other.



$$\vec{\omega}_s = \omega_2 \cos \theta \hat{i} + \omega_2 \sin \theta \hat{j}$$

$$\vec{\omega}_o = \omega_1 \hat{k}$$

$$\theta = \omega_1 t$$

$$\vec{\omega}_{2/1} = \vec{\omega}_s + \vec{\omega}_o$$

$$= \omega_2 \cos \omega_1 t \hat{i} + \omega_2 \sin \omega_1 t \hat{j} + \omega_1 \hat{k}$$

$$|\vec{\omega}_{2/1}| = \sqrt{\omega_2^2 + \omega_1^2}$$

$$\text{Also, } \alpha_{2/1} = \frac{d}{dt} \omega_{2/1}$$

$$= -\omega_2 \omega_1 \sin \omega_1 t + \omega_2 \omega_1 \cos \omega_1 t + 0$$

$$\therefore |\alpha_{2/1}| = \sqrt{\omega_2 \omega_1}$$

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1.56. A solid body rotates with angular velocity $\omega = at\mathbf{i} + bt^2\mathbf{j}$, where $a = 0.50 \text{ rad/s}^2$, $b = 0.060 \text{ rad/s}^3$, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find:

- (a) the moduli of the angular velocity and the angular acceleration at the moment $t = 10.0 \text{ s}$;
 (b) the angle between the vectors of the angular velocity and the angular acceleration at that moment.

$$\omega = at\mathbf{i} + bt^2\mathbf{j} \Rightarrow \alpha = \frac{d\omega}{dt} = a\mathbf{i} + 2bt\mathbf{j}$$

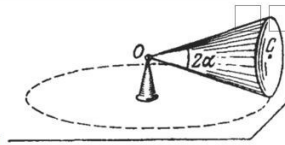
$$|\omega| = \sqrt{(at)^2 + (bt^2)^2}$$

$$|\alpha| = \sqrt{a^2 + (2bt)^2}$$

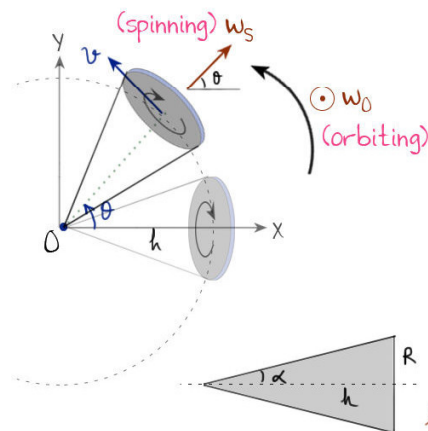
$$\cos\theta = \frac{\vec{\omega} \cdot \vec{\alpha}}{|\omega||\alpha|} \Rightarrow \theta = \cos^{-1} \frac{a^2t + 2b^2t^3}{|\omega||\alpha|}$$

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1.57. A round cone with half-angle $\alpha = 30^\circ$ and the radius of the base $R = 5.0 \text{ cm}$ rolls uniformly and without slipping over a horizontal plane as shown in Fig. 1.8. The cone apex is hinged at the point O which is on the same level with the point C , the cone base centre. The velocity of point C is $v = 10.0 \text{ cm/s}$. Find the moduli of



- (a) the vector of the angular velocity of the cone and the angle it forms with the vertical;
 (b) the vector of the angular acceleration of the cone. Prerequisite: 1.55



$$\omega_0 = \frac{v}{h} = \frac{v}{R \cot \alpha}$$

$$\omega_s = \frac{v}{R} \quad (\text{Rolling})$$

$$\vec{\omega} = \vec{\omega}_0 + \vec{\omega}_s$$

$$= \omega_0 \hat{k} + \omega_s \cos\theta \hat{i} + \omega_s \sin\theta \hat{j}$$

$$\Rightarrow |\vec{\omega}| = \frac{v}{R} \sqrt{\frac{1}{\cot^2 \alpha} + 1} = \frac{v}{R \cos \alpha}$$

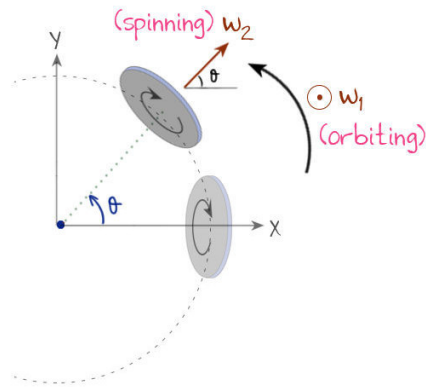
Also,

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = -\omega_s \omega_0 \sin\theta \hat{i} + \omega_s \omega_0 \cos\theta \hat{j}$$

$$\Rightarrow |\vec{\alpha}| = \omega_s \omega_0 = \frac{v^2}{R^2 \cot^2 \alpha} = \frac{v^2 \tan \alpha}{R^2}$$

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1.58. A solid body rotates with a constant angular velocity $\omega_0 = 0.50 \text{ rad/s}$ about a horizontal axis AB . At the moment $t = 0$ the axis AB starts turning about the vertical with a constant angular acceleration $\beta_0 = 0.10 \text{ rad/s}^2$. Find the angular velocity and angular acceleration of the body after $t = 3.5 \text{ s}$. Prerequisite: 1.55



$$\begin{aligned} \vec{\omega} &= \vec{\omega}_{\text{orbit}} + \vec{\omega}_{\text{spin}} \\ &= \vec{\omega}_1 + \vec{\omega}_2 \\ &= \omega_1 \hat{k} + \omega_0 \cos \theta \hat{i} + \omega_0 \sin \theta \hat{j} \\ &\quad \downarrow \quad \quad \quad \downarrow \\ &\quad \beta_0 t \quad \quad \quad \frac{1}{2} \beta_0 t^2 \end{aligned}$$

$$\Rightarrow |\vec{\omega}| = \sqrt{(\beta_0 t)^2 + \omega_0^2}$$

$$\begin{aligned} \vec{\alpha} &= \frac{d\vec{\omega}}{dt} = \beta_0 \hat{k} + \omega_0 \left(\frac{\beta_0}{2} t \right) (-\sin \theta) \hat{i} + \omega_0 \frac{\beta_0}{2} t \cos \theta \hat{j} \\ |\vec{\alpha}| &= \sqrt{\beta_0^2 + \beta_0 \omega_0^2 t^2} = \beta_0 \sqrt{1 + \omega_0^2 t^2} \end{aligned}$$

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1.59. An aerostat of mass m starts coming down with a constant acceleration w . Determine the ballast mass to be dumped for the aerostat to reach the upward acceleration of the same magnitude. The air drag is to be neglected.

Let the upward force on the balloon be k (a constant, as it depends on volume)

$$mg - k = mw$$

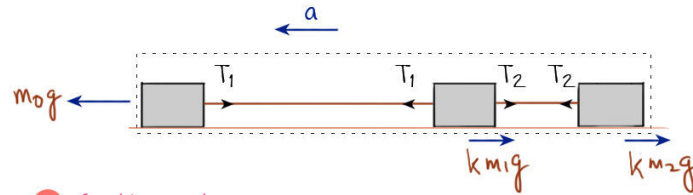
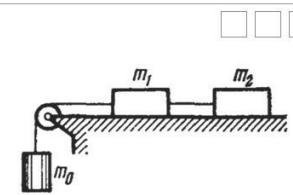
$$k - (m - \Delta m)g = (m - \Delta m)w$$

eliminating k

$$\Delta m = \frac{2mw}{g + w}$$

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1.60. In the arrangement of Fig. 1.9 the masses m_0 , m_1 , and m_2 of bodies are equal, the masses of the pulley and the threads are negligible, and there is no friction in the pulley. Find the acceleration w with which the body m_0 comes down, and the tension of the thread binding together the bodies m_1 and m_2 , if the coefficient of friction between these bodies and the horizontal surface is equal to k . Consider possible cases.



A On the system,

$$m_0 g - k m_1 g - k m_2 g = (m_0 + m_1 + m_2) a$$

$$\Rightarrow a = \frac{(m_0 - k m_1 - k m_2) g}{m_0 + m_1 + m_2}$$

B On block 2:

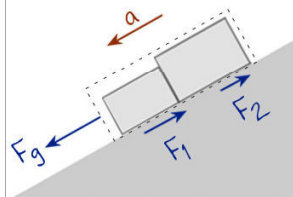
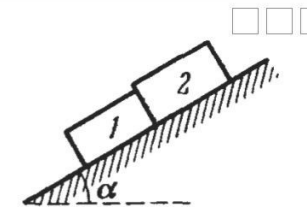
$$T_2 - k m_2 g = m_2 a$$

$$\Rightarrow T_2 = m_2 a + k m_2 g$$

$$= \frac{(k+1) m_0 m_2 g}{m_0 + m_1 + m_2}$$

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1.61. Two touching bars 1 and 2 are placed on an inclined plane forming an angle α with the horizontal (Fig. 1.10). The masses of the bars are equal to m_1 and m_2 , and the coefficients of friction between the inclined plane and these bars are equal to k_1 and k_2 respectively, with $k_1 > k_2$. Find:



A On the system:

$$F_g - F_1 - F_2 = (m_1 + m_2) a$$

$$\Rightarrow (m_1 + m_2) g \sin \alpha - k_1 m_1 g \cos \alpha - k_2 m_2 g \cos \alpha = (m_1 + m_2) a \quad (1)$$

Let the interaction force be F .

On block 1:

$$F_1 g + F - F_1 = m_1 a$$

$$\Rightarrow m_1 g \sin \alpha + F - k_1 m_1 g \cos \alpha = m_1 a \quad (2)$$

$$\text{From 1, 2: } F = \frac{m_1 m_2 \cos \alpha (k_1 - k_2) g}{m_1 + m_2}$$

B For bars to slide: $a > 0$ (3)

From 1, 3:

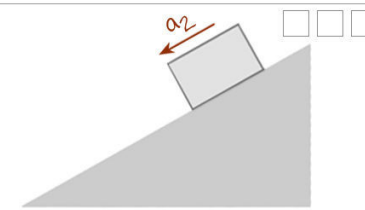
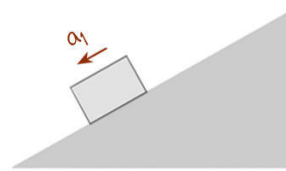
$$(m_1 + m_2) g \sin \alpha > (k_1 m_1 + k_2 m_2) g \cos \alpha$$

$$\Rightarrow \tan \alpha > \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2}$$

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Significance of $k_1 > k_2$

Condition for blocks to slide together: $a_1 < a_2$
(Otherwise block 1 will slide down faster)



$$a_1 < a_2$$

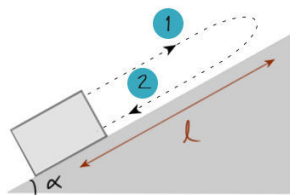
$$\Rightarrow \frac{m_1 g \sin \alpha - k_1 m_1 g \cos \alpha}{m_1} < \frac{m_2 g \sin \alpha - k_2 m_2 g \cos \alpha}{m_2}$$

$$\Rightarrow -k_1 < -k_2$$

$$\Rightarrow k_1 > k_2 //$$

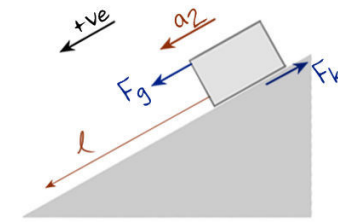
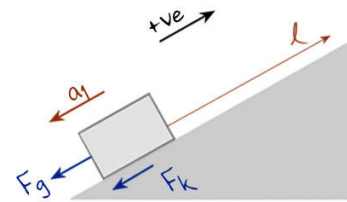
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1.62. A small body was launched up an inclined plane set at an angle $\alpha = 15^\circ$ against the horizontal. Find the coefficient of friction, if the time of the ascent of the body is $\eta = 2.0$ times less than the time of its descent.



$$F_g = mg \sin \alpha$$

$$F_k = k mg \cos \alpha$$



5 equations of motion:

$$v = u + at \quad (1)$$

$$s = ut + \frac{1}{2} at^2 \quad (2)$$

$$v^2 = u^2 + 2as \quad (3)$$

$$1, 2 \Rightarrow s = vt - \frac{1}{2} at^2 \quad (4)$$

$$1, 3 \Rightarrow s = \frac{u+v}{2} t \quad (5)$$

Using 4th equation of motion

$$1 \quad l = vt - \frac{1}{2} at_1^2 \quad (1)$$

$$a_1 = -g \sin \alpha - k g \cos \alpha$$

$$2 \quad l = ut + \frac{1}{2} at_2^2 \quad (2)$$

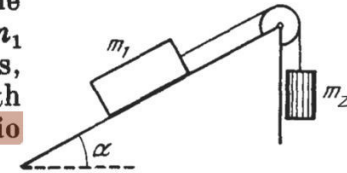
$$a_2 = g \sin \alpha - k g \cos \alpha$$

Dividing 1 and 2,

$$\frac{t_1^2}{t_2^2} = \frac{\sin \alpha - k \cos \alpha}{\sin \alpha + k \cos \alpha} \Rightarrow k = \left(\frac{\eta^2 - 1}{\eta^2 + 1} \right) \tan \alpha //$$

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1.63. The following parameters of the arrangement of Fig. 1.11 are available: the angle α which the inclined plane forms with the horizontal, and the coefficient of friction k between the body m_1 and the inclined plane. The masses of the pulley and the threads, as well as the friction in the pulley, are negligible. Assuming both bodies to be motionless at the initial moment, find the mass ratio



m_2/m_1 at which the body m_2

- (a) starts coming down;
- (b) starts going up;
- (c) is at rest.

A System moves right:

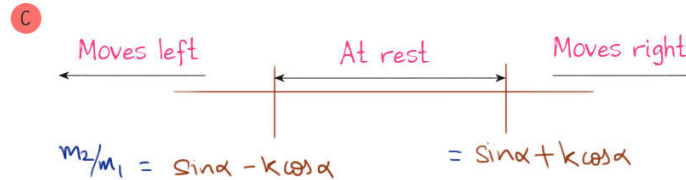
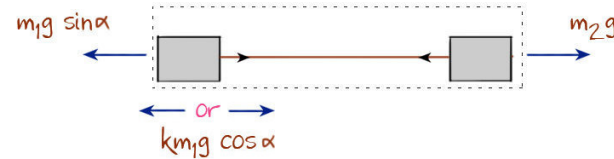
$$m_2g > m_1g \sin \alpha + km_1g \cos \alpha$$

$$\Rightarrow \frac{m_2}{m_1} > \sin \alpha + k \cos \alpha$$

B System moves left:

$$m_1g \sin \alpha > m_2g + km_1g \cos \alpha$$

$$\Rightarrow \frac{m_2}{m_1} < \sin \alpha - k \cos \alpha$$



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1.64. The inclined plane of Fig. 1.11 forms an angle $\alpha = 30^\circ$ with the horizontal. The mass ratio $m_2/m_1 = \eta = 2/3$. The coefficient of friction between the body m_1 and the inclined plane is equal to $k = 0.10$. The masses of the pulley and the threads are negligible. Find the magnitude and the direction of acceleration of the body m_2 when the formerly stationary system of masses starts moving.

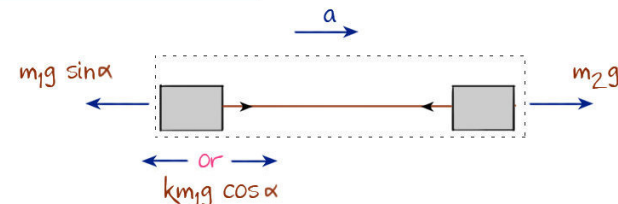
Finding the direction of friction by comparing forces in absence of friction:

$$m_1g \sin \alpha \bigcirc m_2g$$

$$\sin 30^\circ \bigcirc \eta$$

$$\frac{1}{2} \bigcirc \frac{2}{3}$$

∴ System will move to the right



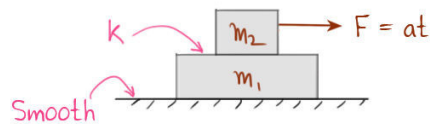
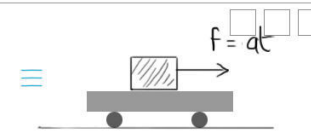
Hence,

$$a = \frac{m_2g - m_1g \sin \alpha - km_1g \cos \alpha}{m_1 + m_2}$$

$$= \frac{g(\eta - \sin \alpha - k \cos \alpha)}{1 + \eta} \quad (\text{Towards right})$$

Youtube / Arihant Senior

1.65. A plank of mass m_1 with a bar of mass m_2 placed on it lies on a smooth horizontal plane. A horizontal force growing with time t as $F = at$ (a is constant) is applied to the bar. Find how the accelerations of the plank w_1 and of the bar w_2 depend on t , if the coefficient of friction between the plank and the bar is equal to k . Draw the approximate plots of these dependences.



Initially both will move together.
Let's say common acceleration is a_c

$$(m_1 + m_2) a_c = at$$

$$\Rightarrow a_c = \frac{at}{m_1 + m_2}$$

Friction will increase with time until it reaches its maximum value.

Blocks will move together as long as...

$$m_1 a_c < \text{maximum frictional force}$$

$$\text{i.e. } m_1 a_c < k m_2 g$$

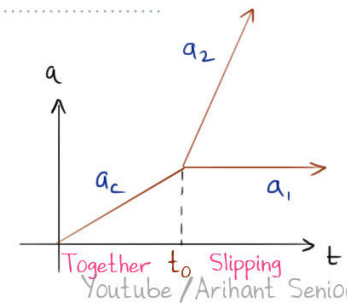
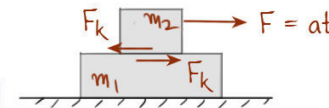
$$\text{i.e. } t < \frac{k m_2 g (m_1 + m_2)}{m_1 a} \rightarrow t_0$$

Before slipping, both blocks: $a_c = \frac{at}{m_1 + m_2}$

After slipping (after t_0):

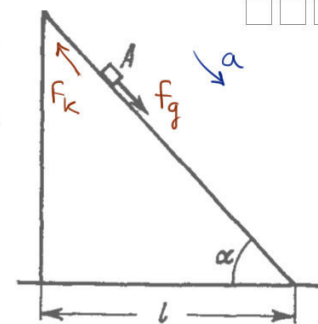
$$\begin{aligned} \text{① } m_1 a_1 &= k m_2 g & \text{② } m_2 a_2 &= at - k m_2 g \\ a_1 &= \frac{k m_2 g}{m_1} & a_2 &= \frac{a}{m_2} t - kg \end{aligned}$$

(constant)



Youtube / Arihant Senior

1.66. A small body A starts sliding down from the top of a wedge (Fig. 1.12) whose base is equal to $l = 2.10$ m. The coefficient of friction between the body and the wedge surface is $k = 0.140$. At what value of the angle α will the time of sliding be the least? What will it be equal to?



$$ma = F_g - F_k = mg \sin \alpha - k mg \cos \alpha$$

$$\Rightarrow a = g(\sin \alpha - k \cos \alpha)$$

To cover the slope of length $\frac{l}{\cos \alpha}$

$$\begin{aligned} \frac{l}{\cos \alpha} &= \frac{1}{2} at^2 \\ &= \frac{1}{2} g (\sin \alpha - k \cos \alpha) t^2 \end{aligned}$$

$$\Rightarrow t^2 = \frac{2l}{ag (\sin \alpha - k \cos \alpha) \cos \alpha}$$

For minimum time, denominator maximum,

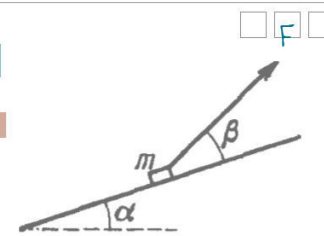
$$\therefore \frac{d}{d\alpha} [(\sin \alpha - k \cos \alpha) \cos \alpha] = 0$$

$$\Rightarrow \cos 2\alpha + k \sin 2\alpha = 0$$

$$\Rightarrow \tan 2\alpha = \frac{-1}{k}$$

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1.67. A bar of mass m is pulled by means of a thread up an inclined plane forming an angle α with the horizontal (Fig. 1.13). The coefficient of friction is equal to k . Find the angle β which the thread must form with the inclined plane for the tension of the thread to be minimum. What is it equal to?



For minimum tension (i.e. F), while friction is maximum, acceleration must be 0. i.e. net force must be zero.

∴ Along the plane:

$$F \cos \beta = mg \sin \alpha + k(mg \cos \alpha - F \sin \beta)$$

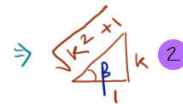
$$\Rightarrow F = \frac{mg(\sin \alpha + k \cos \alpha)}{\cos \beta + k \sin \beta} \quad (1)$$

$F_{\min}$ at given β

Needs to be maximum for overall minimum F

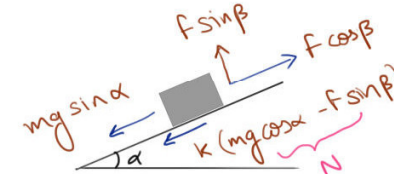
$$\therefore \frac{d}{d\beta}(\cos \beta + k \sin \beta) = 0$$

$$\Rightarrow \tan \beta = k$$



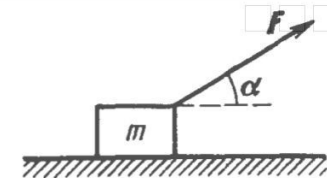
$$\text{From 1, 2: } F_{\min} = \frac{mg(\sin \alpha + k \cos \alpha)}{\frac{1}{\sqrt{k^2 + 1}} + k \cdot \frac{k}{\sqrt{k^2 + 1}}} = \frac{mg(\sin \alpha + k \cos \alpha)}{\sqrt{k^2 + 1}}$$

$F_{\min} \text{ overall}$



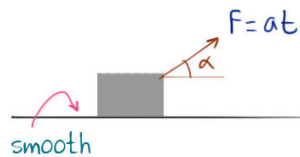
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1.68. At the moment $t = 0$ the force $F = at$ is applied to a small body of mass m resting on a smooth horizontal plane (a is a constant). The permanent direction of this force forms an angle α with the horizontal (Fig. 1.14). Find:



(a) the velocity of the body at the moment of its breaking off the plane;

(b) the distance traversed by the body up to this moment.



Lets say it breaks contact at t_b

When it breaks: $F_{\text{up}} = mg$

$$\Rightarrow at_b \sin \alpha = mg$$

$$\Rightarrow t_b = \frac{mg}{a \sin \alpha}$$

$$F = ma \text{ or}$$

$$F dt = m dv \text{ (impulse} = \Delta p)$$

$$\Rightarrow F \cos \alpha = m \frac{dv}{dt}$$

$$\Rightarrow \int_0^{t_b} at \cos \alpha dt = \int_0^v m dv$$

$$\Rightarrow v = \frac{\cos \alpha}{2m} at^2$$

$$v_b = \frac{mg^2 \cos \alpha}{2a \sin^2 \alpha}$$

$$\frac{ds}{dt} = \frac{\cos \alpha}{2m} at^2$$

$$\Rightarrow \int_0^{s_b} ds = \frac{a \cos \alpha}{2m} \int_0^{t_b} t^2 dt$$

$$s_b = \frac{a \cos \alpha}{2m} \frac{t_b^3}{3}$$

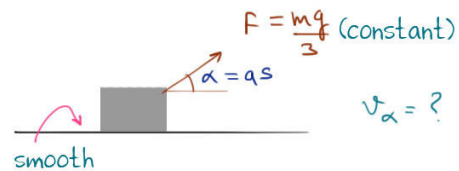
$$= \frac{m^2 g^3 \cos \alpha}{6a^2 \sin^3 \alpha}$$

$$v_{\text{break}} = ?$$

$$s_{\text{break}} = ?$$

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1.69. A bar of mass m resting on a smooth horizontal plane starts moving due to the force $F = mg/3$ of constant magnitude. In the process of its rectilinear motion the angle α between the direction of this force and the horizontal varies as $\alpha = as$, where a is a constant, and s is the distance traversed by the bar from its initial position. Find the velocity of the bar as a function of the angle α .



Along plane $F = ma$:

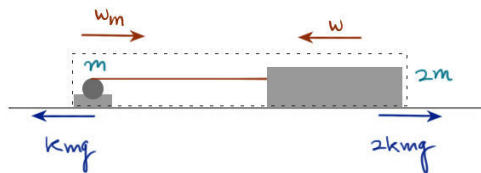
$$F \cos \alpha = m \left(v \frac{dv}{ds} \right)$$

$$\Rightarrow \int_0^{\alpha} F \cos(as) ds = m \int_0^v v dv$$

$$\Rightarrow \frac{F \sin(as)}{a} = \frac{mv^2}{2} \Rightarrow v = \sqrt{\frac{2g \sin \alpha}{3a}}$$

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1.70. A horizontal plane with the coefficient of friction k supports two bodies: a bar and an electric motor with a battery on a block. A thread attached to the bar is wound on the shaft of the electric motor. The distance between the bar and the electric motor is equal to l . When the motor is switched on, the bar, whose mass is twice as great as that of the other body, starts moving with a constant acceleration w . How soon will the bodies collide?



Let the acceleration of motor be w_m .

Using, $m_1 a_1 + m_2 a_2 = F_{\text{system}}$

$$m w_m - 2m w = 2kmg - kmg$$

$$\Rightarrow w_m = kg + 2w$$

Let time taken to collide be t_0

$$l = \frac{1}{2} (w + w_m) t_0^2$$

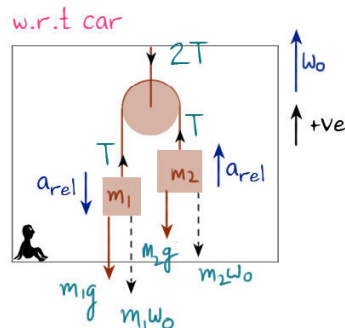
$$\Rightarrow t_0 = \sqrt{\frac{2l}{kg + 3w}}$$

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1.71. A pulley fixed to the ceiling of an elevator car carries a thread whose ends are attached to the loads of masses m_1 and m_2 . The car starts going up with an acceleration w_0 . Assuming the masses of the pulley and the thread, as well as the friction, to be negligible find:

(a) the acceleration of the load m_1 relative to the elevator shaft and relative to the car;

(b) the force exerted by the pulley on the ceiling of the car.



$$\begin{aligned} \text{A } m_1 g + m_1 w_0 - T &= m_1 a_{rel} \\ T - m_2 g - m_2 w_0 &= m_2 a_{rel} \end{aligned}$$

Adding,

$$\Rightarrow a_{rel} = \frac{(m_1 - m_2)(g + w_0)}{(m_1 + m_2)}$$

In vector form

$$- \vec{a}_{rel} = \frac{(m_1 - m_2)(-\vec{g} + \vec{w}_0)}{m_1 + m_2} //$$

$$\Rightarrow \vec{a}_{1/g} = \frac{(m_1 - m_2)\vec{g} + 2m_2\vec{w}_0}{m_1 + m_2}$$

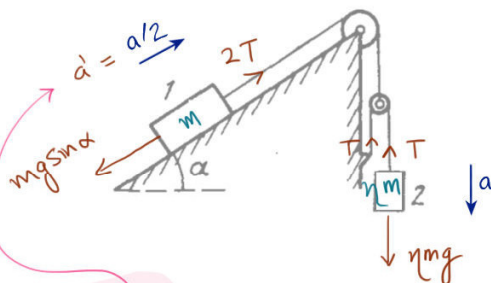
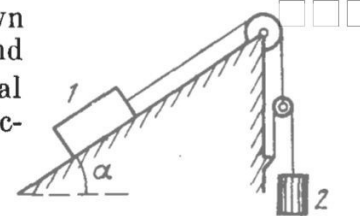
$$\begin{aligned} \text{B } \vec{T} + m_1 \vec{g} &= m_1 \vec{a}_{1/g} \\ \Rightarrow \vec{T} &= m_1 (\vec{a}_{1/g} - \vec{g}) \\ &= \frac{2m_1 m_2}{m_1 + m_2} (-\vec{g} + \vec{w}_0) \end{aligned}$$

Force by pulley on ceiling =

$$-2\vec{T} = \frac{4m_1 m_2}{m_1 + m_2} (\vec{g} - \vec{w}_0)$$

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1.72. Find the acceleration w of body 2 in the arrangement shown in Fig. 1.15, if its mass is η times as great as the mass of bar 1 and the angle that the inclined plane forms with the horizontal is equal to α . The masses of the pulleys and the threads, as well as the friction, are assumed to be negligible. Look into possible cases.



$$\begin{aligned} a' &= \frac{0 + a}{2} \\ \Rightarrow a' &= a/2 \end{aligned}$$

$$(\eta mg - T = \eta ma) \times 2$$

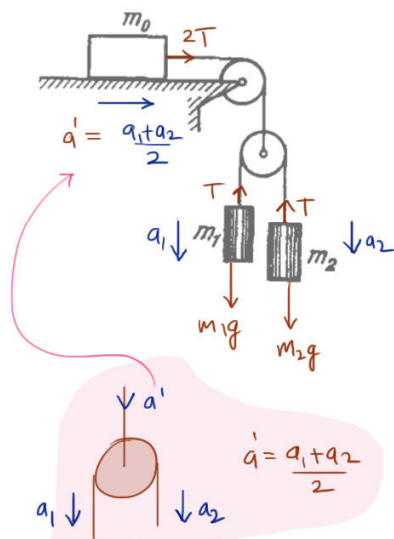
$$2T - mg \sin \alpha = \frac{ma}{2}$$

$$\text{Adding: } 2\eta mg - mg \sin \alpha = 2\eta ma + \frac{ma}{2}$$

$$\Rightarrow a = \frac{2g(2\eta - \sin \alpha)}{(\leftarrow \eta + 1) //}$$

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1.73. In the arrangement shown in Fig. 1.16 the bodies have masses m_0, m_1, m_2 , the friction is absent, the masses of the pulleys and the threads are negligible. Find the acceleration of the body m_1 . Look into possible cases.

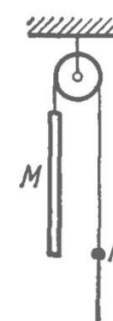


$$\begin{aligned} m_1 g - T &= m_1 a_1 \quad (1) \\ m_2 g - T &= m_2 a_2 \quad (2) \\ 2T &= m_0 \left(\frac{a_1 + a_2}{2} \right) \quad (3) \end{aligned}$$

$$\text{Solving, } a_1 = \frac{4m_1 m_2 + m_0(m_1 - m_2)}{4m_1 m_2 + m_0(m_1 + m_2)} g //$$

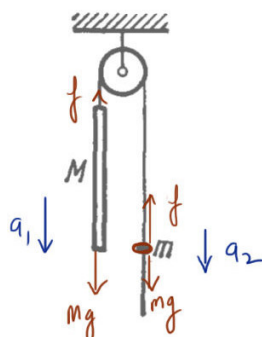
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1.74. In the arrangement shown in Fig. 1.17 the mass of the rod M exceeds the mass m of the ball. The ball has an opening permitting it to slide along the thread with some friction. The mass of the pulley and the friction in its axle are negligible. At the initial moment the ball was located opposite the lower end of the rod. When set free, both bodies began moving with constant accelerations. Find the friction force between the ball and the thread if t seconds after the beginning of motion the ball got opposite the upper end of the rod. The rod length equals l .



Let the friction on ring be f .
Relative to rope, ring slides down so f 's direction will be upwards

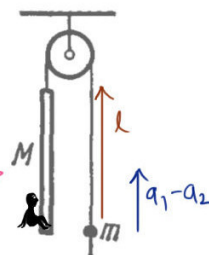
Ring is pulled up by $f \Rightarrow$ Rope is pulled down by $f =$ Tension in the rope
(Action reaction)



$$Mg - f = M a_1 \quad (1)$$

$$mg - f = m a_2 \quad (2)$$

$$\begin{aligned} \text{w.r.t } M: & \quad l = \frac{1}{2} (a_1 - a_2) t^2 \quad (3) \end{aligned}$$

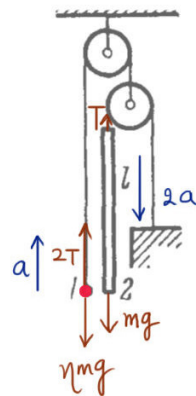
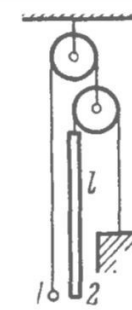


From 1,2,3:

$$f = \frac{2lMm}{(M-m)t^2} //$$

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1.75. In the arrangement shown in Fig. 1.18 the mass of ball 1 is $\eta = 1.8$ times as great as that of rod 2. The length of the latter is $l = 100$ cm. The masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod?

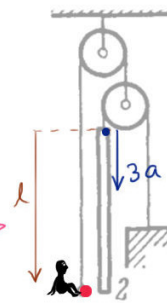


$$2T - \eta mg = \eta ma \quad (1)$$

$$mg - T = m 2a \quad (2)$$

w.r.t M: $\longrightarrow$

$$l = \frac{1}{2} 3a t^2 \quad (3)$$

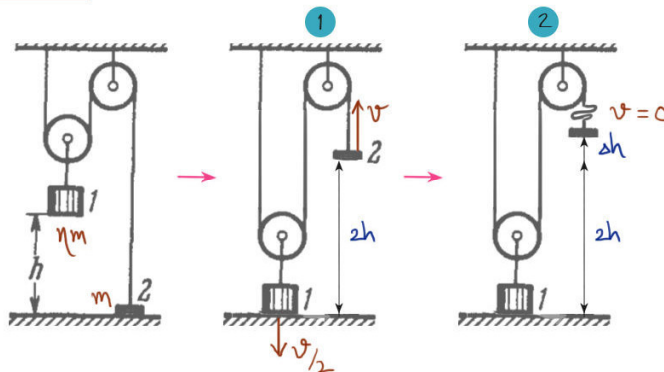


From 1,2,3:

$$t = \sqrt{\frac{2l(\eta+4)}{3(2-\eta)g}}$$

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1.76. In the arrangement shown in Fig. 1.19 the mass of body 1 is $\eta = 4.0$ times as great as that of body 2. The height $h = 20$ cm. The masses of the pulleys and the threads, as well as the friction, are negligible. At a certain moment body 2 is released and the arrangement set in motion. What is the maximum height that body 2 will go up to?



Just before 1 touches ground

When 2 reaches its maximum height.

1 Energy conservation:

$$\eta mgh - mg(2h) = \frac{1}{2} \eta m \left(\frac{v}{2}\right)^2 + \frac{1}{2} m v^2$$

$$\Rightarrow (\eta - 2)mgh = \frac{1}{2} m v^2 \left(1 + \frac{\eta}{4}\right)$$

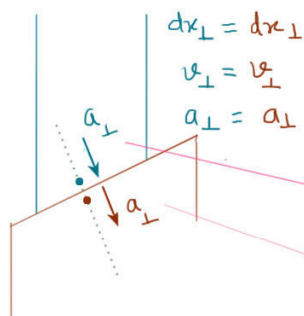
$$\Rightarrow v^2 = \frac{8(\eta - 2)gh}{(\eta + 4)}$$

$$2 \Delta h = \frac{v^2}{2g} = \frac{4(\eta - 2)h}{(\eta + 4)}$$

$$\text{Net height} = 2h + \Delta h = \frac{6\eta h}{\eta + 4}$$

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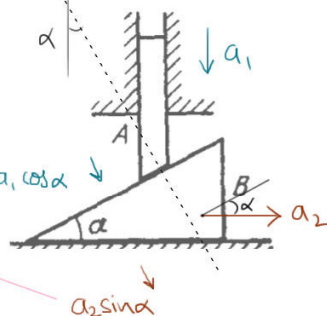
1.77. Find the accelerations of rod A and wedge B in the arrangement shown in Fig. 1.20 if the ratio of the mass of the wedge to that of the rod equals η , and the friction between all contact surfaces is negligible.



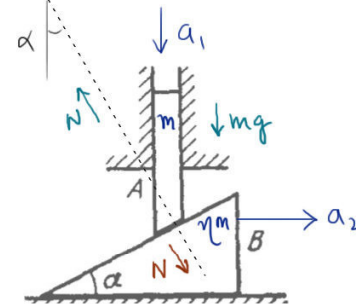
Concept:

When two objects are in contact: displacement/velocity/acceleration in direction perpendicular to common surface is equal. (Otherwise they will separate or 'merge' into each other)

Sol:



$$a_1 \cos \alpha = a_2 \sin \alpha \quad (1)$$



$$mg - N \cos \alpha = ma_1 \quad (2)$$

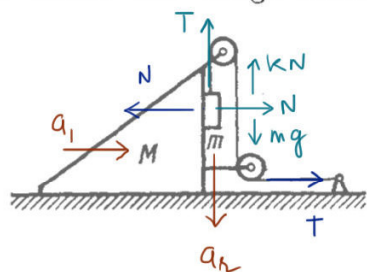
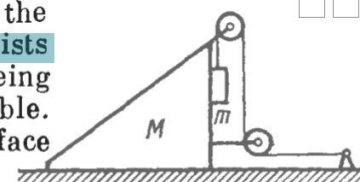
$$N \sin \alpha = \eta m a_2 \quad (3)$$

From 1,2,3:

$$a_1 = \frac{g}{1 + \eta \cot^2 \alpha}, \quad a_2 = \frac{g}{\tan \alpha + \eta \cot \alpha}$$

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1.78. In the arrangement shown in Fig. 1.21 the masses of the wedge M and the body m are known. The appreciable friction exists only between the wedge and the body m , the friction coefficient being equal to k . The masses of the pulley and the thread are negligible. Find the acceleration of the body m relative to the horizontal surface on which the wedge slides.



Let the acceleration of m wrt M be a_r

$$m \text{ in } Y \text{ direction } mg - T - kN = m a_{ry} \quad (1)$$

$$m \text{ in } X \text{ direction } N = m a_{rx} \quad (2)$$

$$M \text{ in } X \text{ direction } T - N = M a_1 \quad (3)$$

$$a_1 = a_r \quad (4)$$

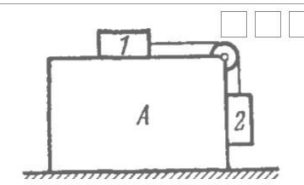
From 1,2,3,4:

$$a_1 = \frac{g}{2 + k + \frac{M}{m}}$$

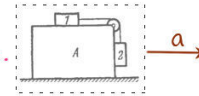
$$\begin{aligned} a_m &= a_{m/M} + a_M \\ &= \vec{a}_r + \vec{a}_1 \\ &= \sqrt{a_r^2 + a_1^2} \\ &= \sqrt{2} a_1 \\ &= \sqrt{2} \frac{g}{2 + k + \frac{M}{m}} \end{aligned}$$

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1.79. What is the minimum acceleration with which bar A (Fig. 1.22) should be shifted horizontally to keep bodies 1 and 2 stationary relative to the bar? The masses of the bodies are equal, and the coefficient of friction between the bar and the bodies is equal to k . The masses of the pulley and the threads are negligible, the friction in the pulley is absent.

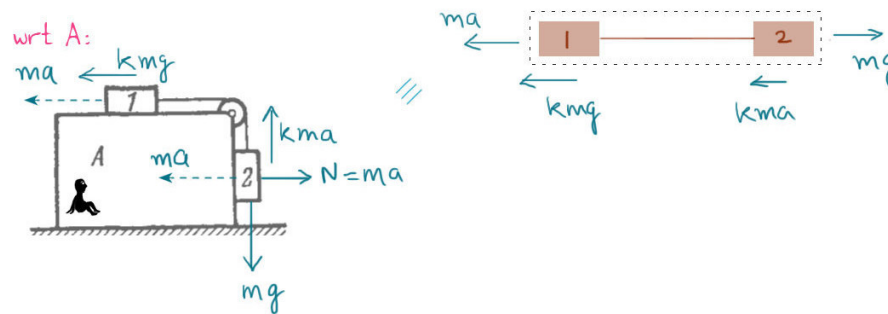


Let the acceleration of bodies be a towards right.



Blocks 1 and 2 have tendency to slip towards right and down respectively.

∴ Frictional forces on them will be towards left and up.



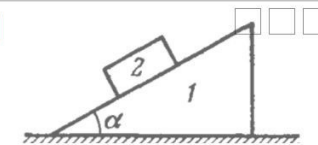
∴ 1 and 2 are at rest wrt A .

$$mg = ma + kmg + kma$$

$$\Rightarrow a = \frac{g(1-k)}{(1+k)}$$

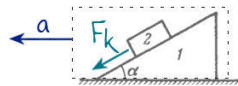
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1.80. Prism 1 with bar 2 of mass m placed on it gets a horizontal acceleration w directed to the left (Fig. 1.23). At what maximum value of this acceleration will the bar be still stationary relative to the prism, if the coefficient of friction between them $k < \cot \alpha$?



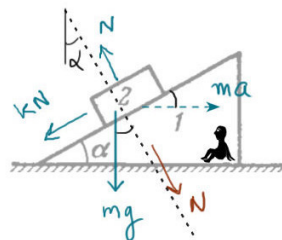
As horizontal acceleration to the left is increased, 2 will have a tendency to slip up the incline.

∴ Friction on it act down the incline.



At maximum acceleration of wedge, friction will be maximum. i.e. kN .

∴ wrt 1:



∴ 2 is at rest wrt 1

$$mg \sin \alpha + kN = ma \cos \alpha \quad (1)$$

$$N = mg \cos \alpha + ma \sin \alpha \quad (2)$$

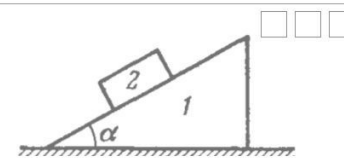
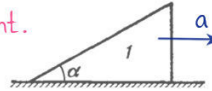
From 1,2:

$$a = g \left(\frac{1 + k \cot \alpha}{\cot \alpha - k} \right)$$

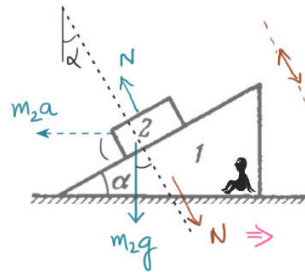
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1.81. Prism 1 of mass m_1 and with angle α (see Fig. 1.23) rests on a horizontal surface. Bar 2 of mass m_2 is placed on the prism. Assuming the friction to be negligible, find the acceleration of the prism.

Let the acceleration of 1 be a towards right.



wrt 1:



$$F_{2\perp} = 0 \Rightarrow N + m_2 a \sin \alpha = m_2 g \cos \alpha \quad (1)$$

$$N \sin \alpha = m_1 a \quad (2)$$

From 1,2:

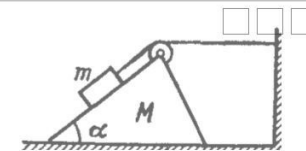
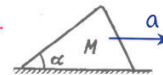
$$a = \frac{g \sin \alpha \cos \alpha}{\frac{M}{m} + \sin^2 \alpha}$$

Youtube / Arihant Senior

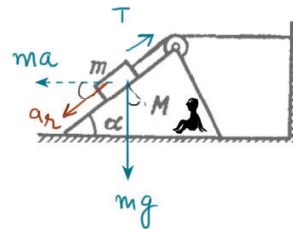
1.82. In the arrangement shown in Fig. 1.24 the masses m of the bar and M of the wedge, as well as the wedge angle α , are known. The masses of the pulley and the thread are negligible. The friction is absent. Find the acceleration of the wedge M .

Let the acceleration of M be a towards right.

Constrained relation: $a = a_r \quad (1)$

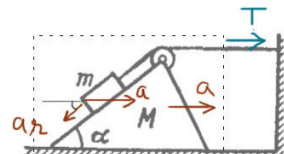


wrt 1:



$$mg \sin \alpha + m a \cos \alpha - T = m a_r \quad (2)$$

System:



$$T = M a + m (a - a_r \cos \alpha) \quad (3)$$

From 1,2,3:

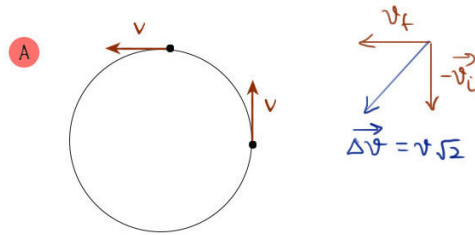
$$a = \frac{g \sin \alpha \cos \alpha}{\frac{M}{m} + \sin^2 \alpha}$$

Youtube / Arihant Senior

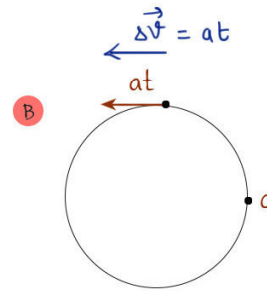
1.83. A particle of mass m moves along a circle of radius R . Find the modulus of the average vector of the force acting on the particle over the distance equal to a quarter of the circle, if the particle moves

- uniformly with velocity v ;
- with constant tangential acceleration w_t , the initial velocity being equal to zero.

$$\vec{F} = \frac{\Delta \vec{p}}{\Delta t} = m \frac{\Delta \vec{v}}{\Delta t} = \frac{m(\vec{v}_f - \vec{v}_i)}{\Delta t}$$



$$F = m \frac{\Delta v}{\Delta t} = \frac{m v \sqrt{2}}{\left(\frac{2\pi R}{4}\right)/v} = \frac{2\sqrt{2} m v^2}{\pi R}$$



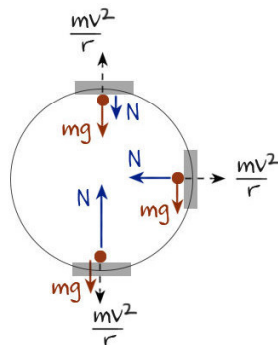
$$F = m \frac{\Delta v}{\Delta t} = \frac{m a t}{t} = m a$$

Youtube / Arihant Senior

1.84. An aircraft loops the loop of radius $R = 500$ m with a constant velocity $v = 360$ km per hour. Find the weight of the flyer of mass $m = 70$ kg in the lower, upper, and middle points of the loop.

Apparent weight: i.e normal reaction from the seat.

w.r.t plane



w.r.t plane, pilot is at rest.

At lower point: $N = mg + \frac{mv^2}{R}$

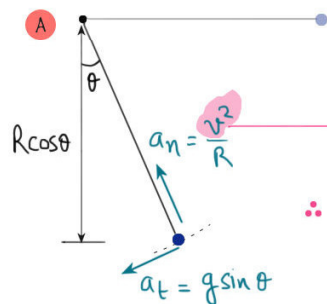
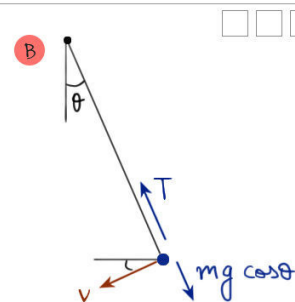
At middle point: $N = \frac{mv^2}{R}$

At upper point: $N = \frac{mv^2}{R} - mg$

Youtube / Arihant Senior

1.85. A small sphere of mass m suspended by a thread is first taken aside so that the thread forms the right angle with the vertical and then released. Find:

- the total acceleration of the sphere and the thread tension as a function of θ , the angle of deflection of the thread from the vertical;
- the thread tension at the moment when the vertical component of the sphere's velocity is maximum;
- the angle θ between the thread and the vertical at the moment when the total acceleration vector of the sphere is directed horizontally.



Energy conservation

$$\frac{1}{2}mv^2 = mgR \cos \theta \quad (1)$$

$$\begin{aligned} \therefore a_{\text{total}} &= \sqrt{a_n^2 + a_t^2} \\ &= \sqrt{(2g \cos \theta)^2 + (g \sin \theta)^2} \\ &= g \sqrt{3 \cos^2 \theta + 1} \quad //, T \end{aligned}$$

$$\text{From 1, } v = \sqrt{2gR \cos \theta}$$

$$\begin{aligned} \Rightarrow v_{\text{vertical}} &= v \sin \theta \\ &= \sqrt{2gR \cos \theta} \sin \theta \end{aligned}$$

∴ For maximum vertical velocity:

$$\frac{d}{d\theta} (\cos \theta \sin^2 \theta) = 0 \quad \text{solving} \rightarrow \tan \theta = \sqrt{2} \quad (2)$$

Also,

$$T - mg \cos \theta = m a_n = m \frac{v^2}{R} = 2mg \cos \theta$$

$$\Rightarrow T = 3mg \cos \theta = 3mg \left(\frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}} mg //$$

Youtube / Arihant Senior

(c) the angle θ between the thread and the vertical at the moment when the total acceleration vector of the sphere is directed horizontally.

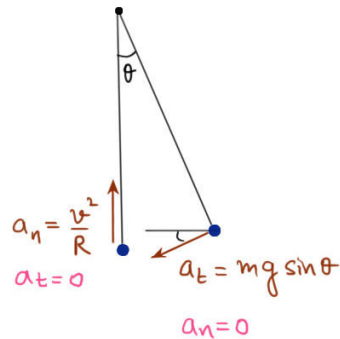
(c) In part B we found,

$$v_y \text{ is maximum at: } \tan \theta = \sqrt{2}$$

$$\Rightarrow \text{at } \tan \theta = \sqrt{2}, \quad \frac{dv_y}{dt} = 0 \Rightarrow a_y = 0$$

$\Rightarrow$ Acceleration is directed horizontally.

1.86. A ball suspended by a thread swings in a vertical plane so that its acceleration values in the extreme and the lowest position are equal. Find the thread deflection angle in the extreme position.



$$g \sin \theta = \frac{v^2}{R} \quad \text{---} \quad \frac{1}{2} m v^2 = m g R (1 - \cos \theta)$$

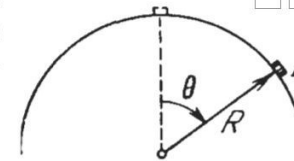
$$\Rightarrow g \sin \theta = 2g(1 - \cos \theta)$$

$$\Rightarrow 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 4 \sin^2 \frac{\theta}{2}$$

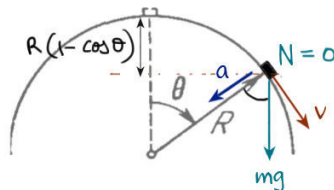
$$\Rightarrow \tan \frac{\theta}{2} = \frac{1}{2} \quad \Rightarrow \quad \theta = 53^\circ$$

Youtube / Arihant Senior

1.87. A small body A starts sliding off the top of a smooth sphere of radius R . Find the angle θ (Fig. 1.25) corresponding to the point at which the body breaks off the sphere, as well as the break-off velocity of the body.



When body breaks off the sphere, Normal reaction becomes zero.



Force = ma

$$mg \cos \theta = \frac{mv^2}{R} \quad (1)$$

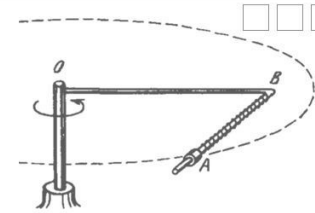
Energy conservation:

$$\frac{1}{2} mv^2 = mgR(1 - \cos \theta) \quad (2)$$

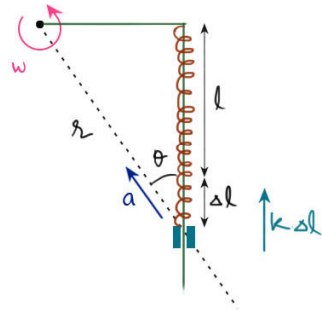
$$\text{From 1,2:} \quad \cos \theta = \frac{2}{3}, \quad v = \sqrt{\frac{2Rg}{3}}$$

Youtube / Arihant Senior

1.88. A device (Fig. 1.26) consists of a smooth L-shaped rod located in a horizontal plane and a sleeve A of mass m attached by a weightless spring to a point B . The spring stiffness is equal to k . The whole system rotates with a constant angular velocity ω about a vertical axis passing through the point O . Find the elongation of the spring. How is the result affected by the rotation direction?



$$\downarrow \frac{\Delta l}{l} = ?$$



In Y direction, only force is spring force.

$$F_y = ma_y$$

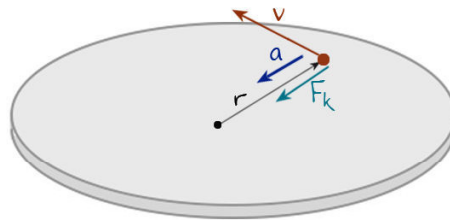
$$\therefore k \Delta l = m \omega^2 \underbrace{r \cos \theta}_{(l + \Delta l)}$$

$$\Rightarrow k \Delta l = m \omega^2 (l + \Delta l)$$

$$\Rightarrow \frac{\Delta l}{l} = \frac{m \omega^2}{k - m \omega^2}$$

Youtube / Arihant Senior

1.89. A cyclist rides along the circumference of a circular horizontal plane of radius R , the friction coefficient being dependent only on distance r from the centre O of the plane as $k = k_0(1 - r/R)$, where k_0 is a constant. Find the radius of the circle with the centre at the point along which the cyclist can ride with the maximum velocity. What is this velocity?



At maximum velocity without sliding, friction must be at its maximum value = kmg .

$$F_k = ma:$$

$$k \left(\frac{v}{g} \right) = \frac{mv^2}{r}$$

$$k_0 \left(1 - \frac{r}{R} \right) g = \frac{v^2}{r}$$

$$\Rightarrow v^2 = k_0 g r \left(1 - \frac{r}{R} \right)$$

$$\Rightarrow v^2 = k_0 g \left(r - \frac{r^2}{R} \right)$$

($v_{\max}$ at given r)

At maximum velocity

$$\frac{dv^2}{dr} = 0 \Rightarrow 1 - \frac{2r}{R} = 0$$

$$\Rightarrow r = R/2$$

Corresponding v :

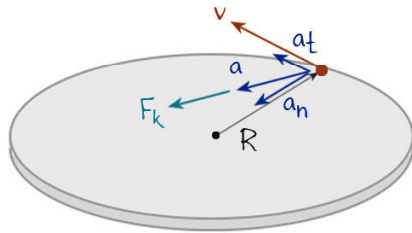
$$v_{\max} = \sqrt{k_0 g \left(\frac{R}{2} - \frac{1}{R} \frac{R^2}{4} \right)} = \sqrt{\frac{k_0 g R}{4}}$$

($v_{\max}$ overall)

Youtube / Arihant Senior

1.90. A car moves with a constant tangential acceleration $a_t = 0.62 \text{ m/s}^2$ along a horizontal surface circumscribing a circle of radius $R = 40 \text{ m}$. The coefficient of sliding friction between the wheels of the car and the surface is $k = 0.20$. What distance will the car ride without sliding if at the initial moment of time its velocity is equal to zero?

Only external horizontal force on car is friction. That determines the net acceleration of car.
Also, when its about to slide, friction is maximum.



$$F_k = ma$$

$$kmg = m \sqrt{a_t^2 + \left(\frac{v^2}{R}\right)^2}$$

$$k^2 g^2 = a_t^2 + \frac{4a_t^2 s^2}{R^2}$$

$$\Rightarrow s = \frac{R}{2a_t} \sqrt{(k^2 g^2 - a_t^2)}$$

Youtube / Arihant Senior

1.91. A car moves uniformly along a horizontal sine curve $y = a \sin(x/\alpha)$, where a and α are certain constants. The coefficient of friction between the wheels and the road is equal to k . At what velocity will the car ride without sliding?

Only external horizontal force on car is friction. That determines the net acceleration of car.
Also, when its about to slide, friction is maximum.

Car has maximum chances of sliding:

Where friction is maximum on the path.

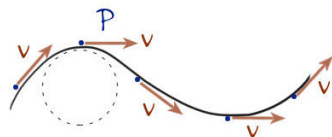
↓

Acceleration is maximum (as only external horizontal force on car is due to friction)

$$\text{(For constant } v: a = \sqrt{a_n^2 + a_t^2} \approx a_n = \frac{v^2}{R_c})$$

↓

Radius of curvature R_c is minimum. Which in sine curve occurs at crests/troughs.

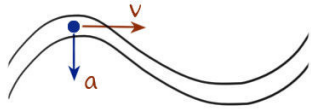


$$\text{At point P: } F_{k_{\max}} = ma_{\max} = \frac{mv_{\max}^2}{R_c}$$

$$\Rightarrow kmg = \frac{mv_{\max}^2}{R_c} \Rightarrow v_{\max} = \sqrt{k g R_c}$$

Youtube / Arihant Senior

Finding value of R_c $R_c = \frac{v^2}{a_\perp}$ 2



At point P, $a_\perp = a_y = \frac{d^2 y}{dt^2} = \frac{d}{dt} \left(\frac{dy}{dt} \right)$

$$y = a \sin\left(\frac{x}{a}\right)$$

$$\begin{aligned} &= \frac{d}{dt} \left(\frac{a}{a} \cos \frac{x}{a} \cdot \frac{dx}{dt} \right) \\ &= \frac{av}{a} \frac{d}{dt} \left(\cos \frac{x}{a} \right) \\ &= \frac{av}{a} \left[-\frac{1}{a} \sin \frac{x}{a} \cdot \frac{dx}{dt} \right] \\ &= -\frac{av^2}{a^2} \quad 3 \end{aligned}$$

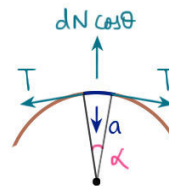
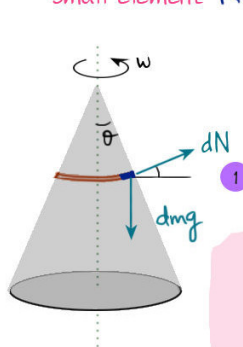
From 2,3 $R_c = \frac{a^2}{a} \rightarrow$ Putting in 1:

$$\Rightarrow v_{\max} = \sqrt{kg R_c} = \sqrt{\frac{kg a^2}{a}}$$

Youtube / Arihant Senior

1.92. A chain of mass m forming a circle of radius R is slipped on a smooth round cone with half-angle θ . Find the tension of the chain if it rotates with a constant angular velocity ω about a vertical axis coinciding with the symmetry axis of the cone.

Calculating vertical and horizontal forces on a small element $R\alpha$ on the chain.



$F = ma$ in horizontal plane:

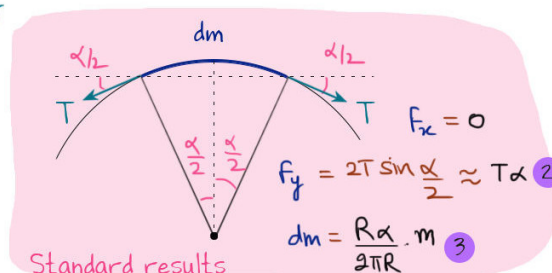
$$dN \sin \theta = dm g \quad 1 \quad (\because F = 0, \text{ vertically})$$

$$T\alpha - dN \cos \theta = dm \omega^2 R \quad 2 \quad \left(\frac{R\alpha}{2\pi R} \right) m \quad 3$$

$$T\alpha = dm \left(\omega^2 R + \frac{g \cos \theta}{\sin \theta} \right)$$

$$T\alpha = \frac{R\alpha}{2\pi R} m \left(\omega^2 R + g \cot \theta \right)$$

$$T = \left[\frac{\omega^2 R + \cot \theta}{g} \right] \frac{mg}{2\pi}$$

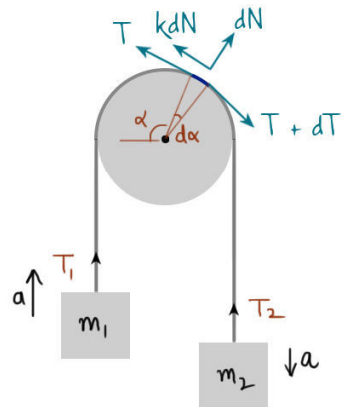


Standard results for F_y due to Tension on a ring element

Youtube / Arihant Senior

1.93. A fixed pulley carries a weightless thread with masses m_1 and m_2 at its ends. There is friction between the thread and the pulley. It is such that the thread starts slipping when the ratio $m_2/m_1 = \eta_0$. Find:

- (a) the friction coefficient;
(b) the acceleration of the masses when $m_2/m_1 = \eta > \eta_0$.



• Thread is massless, net force on it, or any part of it is 0.

Balancing forces on small element:

Normal: $T \frac{d\alpha}{2} + (T + dT) \frac{d\alpha}{2} = dN$

$\Rightarrow T d\alpha = dN$ (2)

Tangential: $dT \cos \frac{d\alpha}{2} = k dN$ (3)

$T_1 - m_1 g = m_1 a$
 $m_2 g - T_2 = m_2 a$ } $\frac{T_2}{T_1} = \frac{m_2 (g - a)}{m_1 (g + a)}$ (1)

From 2,3: $dT = kT d\alpha$

$\Rightarrow \int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\pi k d\alpha$

$\Rightarrow T_2 = T_1 e^{k\pi}$ (4)

From 1,4:

$e^{k\pi} = \eta \frac{(g - a)}{(g + a)}$

Just slipping η_0 $\eta > \eta_0$
 $a = 0$ $a > 0$

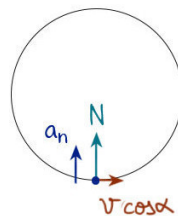
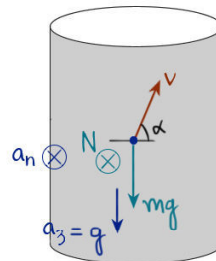
$\Rightarrow e^{k\pi} = \eta_0$ $\Rightarrow e^{k\pi} = \eta \frac{(g - a)}{(g + a)}$

$k = \frac{1}{\pi} \ln \eta_0$

$a = g \frac{(\eta - \eta_0)}{(\eta + \eta_0)}$

Youtube / Arihant Senior

1.94. A particle of mass m moves along the internal smooth surface of a vertical cylinder of radius R . Find the force with which the particle acts on the cylinder wall if at the initial moment of time its velocity equals v_0 and forms an angle α with the horizontal.



$N = m a_n = m \frac{v^2}{R} = \frac{m (v \cos \alpha)^2}{R}$

Youtube / Arihant Senior

1.95. Find the magnitude and direction of the force acting on the particle of mass m during its motion in the plane xy according to the law $x = a \sin \omega t$, $y = b \cos \omega t$, where a , b , and ω are constants.

$$\begin{aligned} x &= a \sin \omega t \quad (1) & y &= b \cos \omega t \quad (2) \\ \Rightarrow a_x &= -a \omega^2 \sin \omega t & \Rightarrow a_y &= -b \omega^2 \cos \omega t \\ &= -\omega^2 x & &= -\omega^2 y \end{aligned}$$

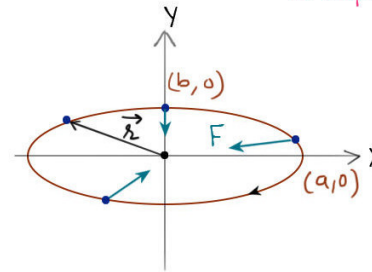
Eliminating t from 1,2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

∴ Elliptical trajectory:

$$\begin{aligned} \vec{F} &= m \vec{a}_x + m \vec{a}_y \\ &= -m \omega^2 (x \hat{i} + y \hat{j}) \\ &= -m \omega^2 \vec{r} \end{aligned}$$

(Central force)

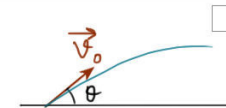


Youtube / Arihant Senior

1.96. A body of mass m is thrown at an angle to the horizontal with the initial velocity $\vec{v}_0$. Assuming the air drag to be negligible, find:

(a) the momentum increment $\Delta \vec{p}$ that the body acquires over the first t seconds of motion;

(b) the modulus of the momentum increment Δp during the total time of motion.



A Method 1

$$\begin{aligned} d\vec{p} &= \vec{F} dt \\ \Rightarrow \Delta \vec{p} &= \vec{F} \Delta t \quad (\because \text{Force is constant}) \\ &= m \vec{g} t \end{aligned}$$

A Method 2

$$\begin{aligned} \Delta \vec{p} &= m (\vec{v}_f - \vec{v}_i) \\ &= m (v_0 \cos \theta \hat{i} + (v_0 \sin \theta - gt) \hat{j}) - m (v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j}) \\ &= m (-gt) \hat{j} \\ &= -mgt \hat{j} \end{aligned}$$

B

$$\begin{aligned} \Delta \vec{p} &= m (\vec{v}_f - \vec{v}_i) \\ &= -m (2 v_0 \sin \theta) \hat{j} = \frac{2m \vec{g} \cdot \vec{v}_0}{g} \hat{j} \end{aligned}$$

$\vec{g} \cdot \vec{v}_0 = g v_0 \cos(90^\circ + \theta) = -g v_0 \sin \theta$

Youtube / Arihant Senior

1.97. At the moment $t = 0$ a stationary particle of mass m experiences a time-dependent force $\vec{F} = a t (\tau - t)$, where a is a constant vector, τ is the time during which the given force acts. Find:

(a) the momentum of the particle when the action of the force discontinued:

(b) the distance covered by the particle while the force acted.

A

$$\vec{F} = \vec{a} t (\tau - t)$$

$$\begin{aligned} \Delta \vec{p} &= \int \vec{F} \cdot dt = \vec{a} \int_0^\tau (\tau t - t^2) dt \\ &= \frac{\vec{a} \tau^3}{6} // \end{aligned}$$

B

$$m \vec{a} = \vec{F}$$

$$m \int_0^\tau d\vec{v} = \vec{a} \int_0^\tau (\tau t - t^2) dt$$

$$\Rightarrow m \vec{v} = \vec{a} \left(\frac{\tau t^2}{2} - \frac{t^3}{3} \right)$$

$$\Rightarrow m \int_0^\tau d\vec{x} = \vec{a} \int_0^\tau \left(\frac{\tau t^2}{2} - \frac{t^3}{3} \right) dt$$

$$\Rightarrow \vec{s} = \frac{\vec{a}}{m} \frac{\tau^4}{12} //$$

Youtube / Arihant Senior

1.98. At the moment $t = 0$ a particle of mass m starts moving due to a force $\vec{F} = F_0 \sin \omega t$, where F_0 and ω are constants. Find the distance covered by the particle as a function of t . Draw the approximate plot of this function.

$$\vec{F} = m \frac{d\vec{v}}{dt} = F_0 \sin \omega t$$

$$\Rightarrow \int_0^t d\vec{v} = \frac{F_0}{m} \int_0^t \sin \omega t dt$$

$$\Rightarrow \vec{v} = \frac{F_0}{m\omega} (1 - \cos \omega t)$$

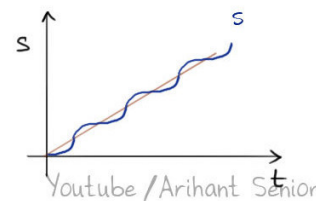
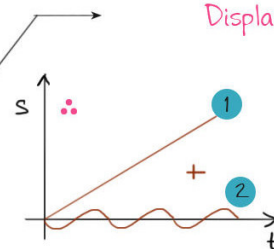
$$\Rightarrow \int_0^t d\vec{x} = \frac{F_0}{m\omega} \int_0^t (1 - \cos \omega t) dt$$

$$\Rightarrow \vec{x} = \frac{F_0}{m\omega} \left(t - \frac{\sin \omega t}{\omega} \right)$$

Always Positive



Displacement = Distance



Youtube / Arihant Senior

1.99. At the moment $t = 0$ a particle of mass m starts moving due to a force $\mathbf{F} = F_0 \cos \omega t$, where F_0 and ω are constants. How long will it be moving until it stops for the first time? What distance will it traverse during that time? What is the maximum velocity of the particle over this distance?

$$F = m \frac{dv}{dt} = F_0 \cos \omega t$$

$$\Rightarrow \int_0^v dv = \frac{F_0}{m} \int_0^t \cos \omega t dt$$

$$\Rightarrow v = \frac{F_0}{m\omega} \sin \omega t$$

It stops first time at $t = \frac{\pi}{\omega}$

$$\Rightarrow \int_0^x dx = \frac{F_0}{m\omega} \int_0^t \sin \omega t dt$$

$$\Rightarrow x = \frac{F_0}{m\omega^2} (1 - \cos \omega t) \xrightarrow{\text{Distance during } \pi/\omega} s = \frac{F_0}{m\omega^2} (1 - \cos \omega \frac{\pi}{\omega}) = \frac{2F_0}{m\omega^2}$$

Maximum velocity $v_{\max} = \frac{F_0}{m\omega}$

Youtube / Arihant Senior

1.100. A motorboat of mass m moves along a lake with velocity v_0 . At the moment $t = 0$ the engine of the boat is shut down. Assuming the resistance of water to be proportional to the velocity of the boat $\mathbf{F} = -r\mathbf{v}$, find:

- how long the motorboat moved with the shutdown engine;
- the velocity of the motorboat as a function of the distance covered with the shutdown engine, as well as the total distance covered till the complete stop;
- the mean velocity of the motorboat over the time interval (beginning with the moment $t = 0$), during which its velocity decreases η times.

$$F = -r v$$

$$m \frac{dv}{dt} = -r v$$

$$\frac{m}{r} \int_{v_0}^v \frac{dv}{v} = \int_0^t dt$$

$$v = v_0 e^{-rt/m} \quad (1)$$

$$m \frac{dv}{dx} = -r v$$

$$\int_{v_0}^v dv = -\frac{r}{m} \int_0^x dx$$

$$v = v_0 - \frac{rx}{m} \quad (2)$$

$$(C) \langle v \rangle = \frac{\Delta x}{\Delta t} = \frac{x}{t} \Big|_{v=v_0/\eta}$$

From 1,2:

$$= \frac{(v_0 - \frac{v_0}{\eta}) \frac{m}{r}}{\frac{m}{r} \ln \eta} = \frac{v_0 (\eta - 1)}{\ln \eta}$$

A From 1:

$$v \rightarrow 0$$

$$\Rightarrow t \rightarrow \infty$$

B From 2:

$$v \rightarrow 0$$

$$\Rightarrow x = mv_0/r$$

Youtube / Arihant Senior

1.101. Having gone through a plank of thickness h , a bullet changed its velocity from v_0 to v . Find the time of motion of the bullet in the plank, assuming the resistance force to be proportional to the square of the velocity.

Let, $f = -kv^2$

$$m \frac{dv}{dt} = -kv^2$$

$$\int_{v_0}^v \frac{dv}{v^2} = -\frac{k}{m} \int_0^t dt$$

$$\frac{1}{v} - \frac{1}{v_0} = \frac{kt}{m} \quad (1)$$

$$m \cdot v \frac{dv}{dx} = -kv^2$$

$$\int_{v_0}^v \frac{dv}{v} = -\frac{k}{m} \int_0^h dx$$

$$\ln \frac{v_0}{v} = \frac{kh}{m}$$

$$k = \frac{m}{h} \ln \frac{v_0}{v} \quad (2)$$

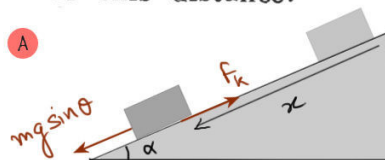
Eliminating k from 1,2

$$t = \frac{h(v_0 - v)}{v_0 v \ln v_0 / v} //$$

Youtube / Arihant Senior

1.102. A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The friction coefficient depends on the distance x covered as $k = ax$, where a is a constant. Find the distance covered by the bar till it stops, and its maximum velocity over this distance.

It will immediately start sliding as friction is 0 at $x=0$.



$$m a_c = mg \sin \alpha - \underbrace{(ax)}_{F_k} mg \cos \alpha \quad (1)$$

$$\int_0^0 v dv = g \int_0^s (\sin \alpha - a \cos \alpha x) dx \quad (2)$$

$$\Rightarrow 0 = \sin \alpha s - a \cos \alpha \frac{s^2}{2}$$

$$\Rightarrow s = \frac{2 \tan \alpha}{a} //$$

B At maximum velocity: $\frac{dv}{dt} = 0$

$$\therefore \text{From 1, } a_c = 0 \Rightarrow \sin \alpha = a x \cos \alpha$$

$$x = \frac{\tan \alpha}{a}$$

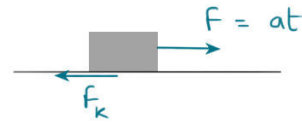
From 2, new limits:

$$\int_0^{v_{\max}} v dv = g \int_0^{\frac{\tan \alpha}{a}} (\sin \alpha - a \cos \alpha x) dx$$

$$\Rightarrow v_{\max} = \sqrt{\frac{g}{a} \sin \alpha \tan \alpha} //$$

Youtube / Arihant Senior

1.103. A body of mass m rests on a horizontal plane with the friction coefficient k . At the moment $t = 0$ a horizontal force is applied to it, which varies with time as $F = at$, where a is a constant vector. Find the distance traversed by the body during the first t seconds after the force action began.



Sliding begins at: $at_0 = kmg$

$$\text{or } t_0 = \frac{km g}{a}$$

$$\therefore x = 0 \quad \left\{ t < \frac{km g}{a} \right\}$$

$$m \frac{dv}{dt} = at - kmg$$

$$\Rightarrow \int_0^v m dv = \int_{t_0}^t a(t - t_0) dt \quad \left\{ t - t_0 \rightarrow \Delta t \right\}$$

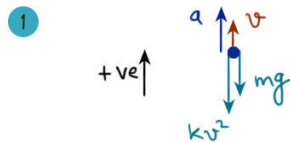
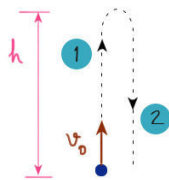
$$\Rightarrow mv = a \frac{\Delta t^2}{2}$$

$$\Rightarrow \int_0^x m dx = \frac{a}{2} \int_0^{\Delta t} \Delta t^2 d\Delta t \Rightarrow x = \frac{a \Delta t^3}{6m}$$

$$\Rightarrow x = \frac{a(t - t_0)^3}{6m} \quad \left\{ \text{For } t > \frac{km g}{a} \right\}$$

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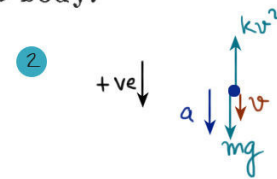
1.104. A body of mass m is thrown straight up with velocity v_0 . Find the velocity v' with which the body comes down if the air drag equals kv^2 , where k is a constant and v is the velocity of the body.



$$m v \frac{dv}{dx} = -(mg + kv^2)$$

$$\Rightarrow -m \int_{v_0}^0 \frac{v dv}{mg + kv^2} = \int_0^h dx$$

$$\Rightarrow h = \frac{m}{2k} \ln \frac{kv_0^2 + mg}{mg}$$



$$m v \frac{dv}{dx} = mg - kv^2$$

$$\int_0^{v'} \frac{m v dv}{mg - kv^2} = \int_0^h dx$$

$$\Rightarrow \ln \frac{mg}{mg - kv'^2} = \frac{2kh}{m}$$

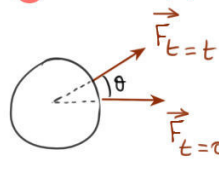
$$\Rightarrow v' = \frac{v_0}{\sqrt{1 + \frac{kv_0^2}{mg}}}$$

Youtube / Arihant Senior

1.105. A particle of mass m moves in a certain plane P due to a force F whose magnitude is constant and whose vector rotates in that plane with a constant angular velocity ω . Assuming the particle to be stationary at the moment $t = 0$, find:

- (a) its velocity as a function of time;
(b) the distance covered by the particle between two successive stops, and the mean velocity over this time.

A Let's assume Force to be in $+X$ direction at $t=0$.



$$\vec{F}_t = F \cos \omega t \hat{i} + F \sin \omega t \hat{j}$$

$$m \frac{d\vec{v}}{dt} = F \cos \omega t \hat{i} + F \sin \omega t \hat{j}$$

$$\Rightarrow m \int d\vec{v} = F \hat{i} \int_0^t \cos \omega t dt + F \hat{j} \int_0^t \sin \omega t dt$$

$$\Rightarrow \vec{v} = \frac{F}{m\omega} \sin \omega t \hat{i} + \frac{F}{m\omega} (1 - \cos \omega t) \hat{j}$$

$$\Rightarrow \text{Speed, } |\vec{v}| = \frac{F}{m\omega} \sqrt{\sin^2 \omega t + (1 - \cos \omega t)^2}$$

$$= \frac{F}{m\omega} \cdot 2 \left| \sin \frac{\omega t}{2} \right| \quad ①$$

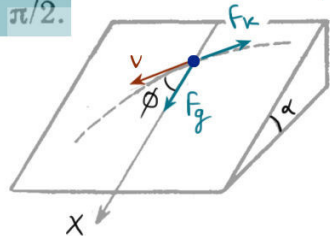
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B From 1, $\int_0^{t_s} ds = \frac{2F}{m\omega} \int_0^{t_s} \sin \frac{\omega t}{2} dt$

Where t_s is when its velocity becomes 0 again.
During 0 to t_s , its velocity is +ve, so we can integrate 1 removing the moduli sign.

$$\therefore s = \frac{2F}{m\omega} \left[-\frac{2}{\omega} \cos \frac{\omega t}{2} \right]_0^{2\pi/\omega} \Rightarrow s = \frac{8F}{m\omega^2} \Rightarrow \langle v \rangle = \frac{s}{t_s} = \frac{8F/m\omega^2}{2\pi/\omega} = \frac{4F}{\pi m\omega}$$

1.106. A small disc A is placed on an inclined plane forming an angle α with the horizontal (Fig. 1.27) and is imparted an initial velocity v_0 . Find how the velocity of the disc depends on the angle ϕ if the friction coefficient $k = \tan \alpha$ and at the initial moment $\phi_0 = \pi/2$.

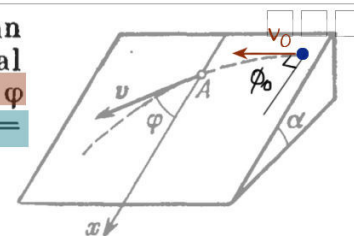


$$F_k = k \underbrace{mg \cos \alpha}_N = mg \sin \alpha$$

$$F_g = mg \sin \alpha$$

$$a_t = \frac{dv}{dt} = \frac{F_g \cos \phi - F_k}{m} = -g \sin \alpha (1 - \cos \phi) \quad ①$$

$$a_n = \frac{dv_n}{dt} = \frac{F_g - F_k \cos \phi}{m} = g \sin \alpha (1 - \cos \phi) \quad ②$$



$$1, 2 \Rightarrow a_t = -a_n$$

$$\frac{dv}{dt} = -\frac{dv_n}{dt}$$

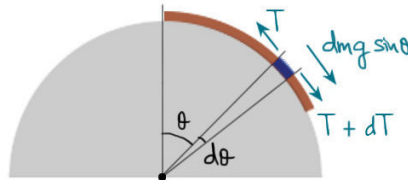
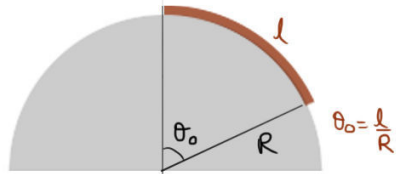
$$\int_{v_0}^v dv = - \int_0^{\phi} v \cos \phi d\phi$$

$$v - v_0 = -v \cos \phi$$

$$\Rightarrow v = \frac{v_0}{1 + \cos \phi}$$

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1.107. A chain of length l is placed on a smooth spherical surface of radius R with one of its ends fixed at the top of the sphere. What will be the acceleration w of each element of the chain when its upper end is released? It is assumed that the length of the chain $l < \frac{1}{2}\pi R$.



$$df_t = dmgsin\theta + dT$$

$$= \frac{Rd\theta}{l} mg \sin\theta + dT$$

$$\int df_t = \frac{mgR}{l} \int_0^{\theta_0} \sin\theta d\theta + \int_0^{\theta_0} dT$$

Net tangential force

$$ma_t = \frac{mgR}{l} (1 - \cos\theta_0)$$

$$\Rightarrow a_t = \frac{gR}{l} (1 - \cos l/R)$$

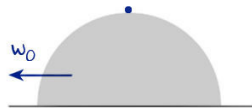
As tension at free end of a rope is always zero.

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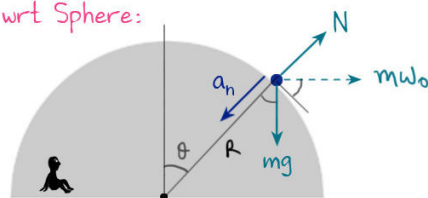
1.108. A small body is placed on the top of a smooth sphere of radius R . Then the sphere is imparted a constant acceleration w_0 in the horizontal direction and the body begins sliding down. Find:

(a) the velocity of the body relative to the sphere at the moment of break-off;

(b) the angle θ_0 between the vertical and the radius vector drawn from the centre of the sphere to the break-off point; calculate θ_0 for $w_0 = g$.



wrt Sphere:



At breakoff, $N=0$.

$$\therefore F = ma$$

$$mg \cos\theta - mw_0 \sin\theta = \frac{mv^2}{R} \quad (1)$$

Energy conservation w.r.t Sphere:

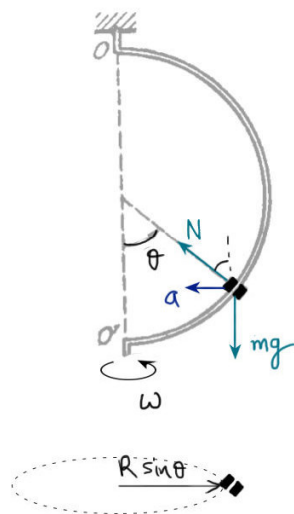
$$mgR(1 - \cos\theta) + mw_0 R \sin\theta = \frac{1}{2}mv^2 \quad (2)$$

$$\text{From 1,2: } v_0 = \sqrt{\frac{2Rg}{3}} \quad \cos\theta = \frac{2 + \eta\sqrt{5+9\eta^2}}{3(1+\eta^2)}$$

$$\eta = \frac{w_0}{g}$$

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1.110. A sleeve A can slide freely along a smooth rod bent in the shape of a half-circle of radius R (Fig. 1.28). The system is set in rotation with a constant angular velocity ω about a vertical axis OO' . Find the angle θ corresponding to the steady position of the sleeve.



$$\begin{aligned} N \cos \theta &= mg \quad (1) \\ N \sin \theta &= ma = m \omega^2 R \sin \theta \quad (2) \end{aligned}$$

↓

$$\left(\frac{mg}{\cos \theta} \right) \sin \theta = m \omega^2 R \sin \theta$$

$$\Rightarrow \sin \theta = 0 \quad \text{or} \quad \cos \theta = \frac{g}{\omega^2 R}$$

$$\text{If } g > \omega^2 R$$

$$\text{If } g < \omega^2 R$$

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1.118. A particle has shifted along some trajectory in the plane xy from point 1 whose radius vector $\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j}$ to point 2 with the radius vector $\mathbf{r}_2 = 2\mathbf{i} - 3\mathbf{j}$. During that time the particle experienced the action of certain forces, one of which being $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$. Find the work performed by the force $\mathbf{F}$. (Here $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{F}$ are given in SI units).

$$\begin{aligned} W &= \vec{F} \cdot d\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1) \quad \left\{ \because \mathbf{F} \text{ is constant} \right\} \\ &= (3\hat{i} + 4\hat{j}) \cdot (\hat{i} - 5\hat{j}) \\ &= 3 - 20 \\ &= -17 \text{ J} \end{aligned}$$

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1.119. A locomotive of mass m starts moving so that its velocity varies according to the law $v = a\sqrt{s}$, where a is a constant, and s is the distance covered. Find the total work performed by all the forces which are acting on the locomotive during the first t seconds after the beginning of motion.

$$v = a\sqrt{s}$$

$$F = ma = m v \frac{dv}{ds} = m(a\sqrt{s}) \frac{a}{2\sqrt{s}} \Rightarrow F = \frac{ma^2}{2} \text{ (Constant)} \Rightarrow a_c = \frac{F}{m} = \frac{a^2}{2}$$

$$\Rightarrow s = \frac{1}{2} a_c t^2 \quad s = \frac{a^2 t^2}{4}$$

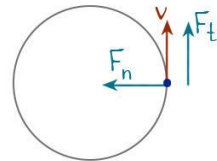
$$W = \vec{F} \cdot \vec{s} = F s = \frac{ma^2}{2} \cdot \frac{a^2 t^2}{4} = \frac{ma^4 t^2}{8} //$$

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1.120. The kinetic energy of a particle moving along a circle of radius R depends on the distance covered s as $T = as^2$, where a is a constant. Find the force acting on the particle as a function of s .

$$\frac{1}{2} mv^2 = as^2$$

$$v = \sqrt{\frac{2a}{m}} s$$



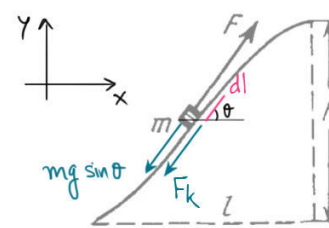
$$F_t = ma = m v \frac{dv}{ds} = m \sqrt{\frac{2a}{m}} s \cdot \frac{\sqrt{2a}}{\sqrt{m}} = 2as$$

$$F_n = \frac{mv^2}{r} = \frac{m}{r} \left(\frac{2a}{m} \right) s^2 = \frac{2as^2}{r}$$

$$\Rightarrow F_{\text{total}} = \sqrt{F_t^2 + F_n^2} = 2as \sqrt{\left(\frac{s}{r}\right)^2 + 1} //$$

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1.121. A body of mass m was slowly hauled up the hill (Fig. 1.29) by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h , the length of its base l , and the coefficient of friction k .



$$\begin{aligned} dW &= F \cdot dl \\ &= (mg \sin \theta + kmg \cos \theta) dl \\ &= mg \frac{dl \sin \theta}{dl} + kmg \frac{dl \cos \theta}{dl} \\ &= mg dy + kmg dx \end{aligned}$$

$$\Rightarrow W = \int dW = mg \int dy + kmg \int dx$$

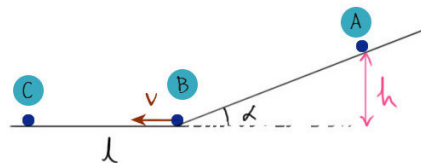
$$\Rightarrow W = mgh + kmg l //$$

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1.122. A disc of mass $m = 50 \text{ g}$ slides with the zero initial velocity down an inclined plane set at an angle $\alpha = 30^\circ$ to the horizontal; having traversed the distance $l = 50 \text{ cm}$ along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient $k = 0.15$ for both inclined and horizontal planes.

For whole process:

$$\begin{aligned} W_{\text{ext}} &= \Delta KE \\ W_g + W_f &= 0 \\ mgh &\Rightarrow W_f = -mgh \end{aligned}$$



$$A \rightarrow B \quad \underbrace{mgh}_{W_g} - \underbrace{kmg \cos \alpha \frac{h}{\sin \alpha}}_{W_f} = \frac{1}{2} mv^2 \quad (1)$$

$$B \rightarrow C \quad \frac{1}{2} mv^2 = kmg l \quad (2)$$

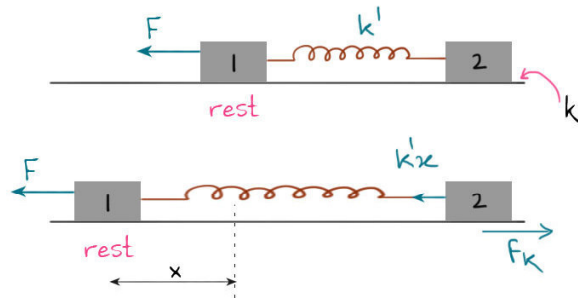
Adding 1,2

$$\Rightarrow mgh = \frac{kmg l}{1 - k \cot \alpha} //$$

$$W_f = -mgh$$

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1.123. Two bars of masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between the bars and the surface is equal to k . What minimum constant force has to be applied in the horizontal direction to the bar of mass m_1 in order to shift the other bar?



For force to be minimum: block 1 will come to rest again when the spring gets the required extension to move block 2.

To move block 2: $k'x = k m_2 g$ (1)

Energy conservation:

$$W_F + W_{\text{friction}} = \Delta PE_{\text{spring}} + \Delta KE$$

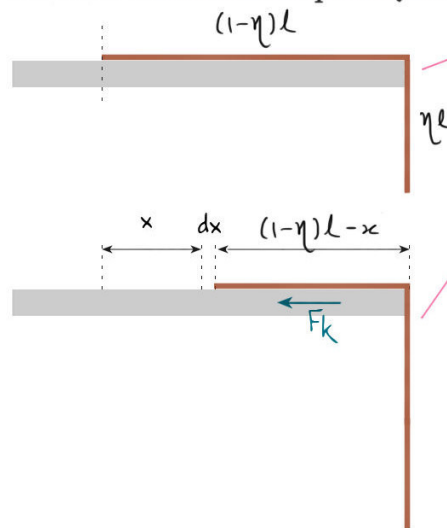
$$F x - (k m_1 g) x = \frac{1}{2} k' x^2$$
 (2)

From 1,2:

$$\Rightarrow F = k \left(m_1 + \frac{m_2}{2} \right) g$$

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1.124. A chain of mass $m = 0.80$ kg and length $l = 1.5$ m rests on a rough-surfaced table so that one of its ends hangs over the edge. The chain starts sliding off the table all by itself provided the overhanging part equals $\eta = 1/3$ of the chain length. What will be the total work performed by the friction forces acting on the chain by the moment it slides completely off the table?



When it starts to slip:

$$\left(\frac{\eta l}{l} m \right) g = k \frac{(1-\eta)l}{l} m g \Rightarrow k = \frac{\eta}{1-\eta}$$
 (1)

While its slipping:

$$dW_k = -F_k dx$$

$$= -k N dx$$

$$\Rightarrow W_k = - \int_0^{(1-\eta)l} k \left[\frac{(1-\eta)l - x}{l} \right] m g dx$$

$$\Rightarrow W_k = -k m g \frac{(1-\eta)^2 l}{2}$$
 (2)

From 1,2:

$$W_k = -\frac{\eta(1-\eta) m g l}{2}$$

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1.125. A body of mass m is thrown at an angle α to the horizontal with the initial velocity v_0 . Find the mean power developed by gravity over the whole time of motion of the body, and the instantaneous power of gravity as a function of time.



For whole motion, Work by gravity = 0.

$$\langle P \rangle = \frac{\Delta W_g}{\Delta t} = 0$$

Instantaneous power = $\vec{F} \cdot \vec{v}$

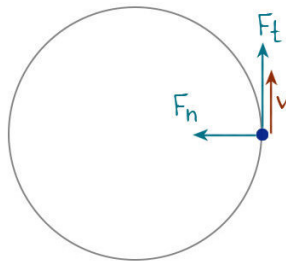
$$F = -mg\hat{j}$$

$$v = v_0 \cos \alpha \hat{i} + (v_0 \sin \alpha - gt) \hat{j}$$

$$\Rightarrow P = \vec{F} \cdot \vec{v} = -mg(v_0 \sin \alpha - gt)$$

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1.126. A particle of mass m moves along a circle of radius R with a normal acceleration varying with time as $a_n = at^2$, where a is a constant. Find the time dependence of the power developed by all the forces acting on the particle, and the mean value of this power averaged over the first t seconds after the beginning of motion.



$$\begin{aligned} P &= \vec{F} \cdot \vec{v} \\ &= (\vec{F}_n + \vec{F}_t) \cdot \vec{v} \\ &= F_t v \\ &= \left(m \frac{dv}{dt}\right) v \end{aligned}$$

$$a_n = at^2$$

$$\frac{v^2}{R}$$

$$\Rightarrow v = \sqrt{aR} t$$

$$= m(\sqrt{aR})\sqrt{aR}t$$

$$= mat$$

$$\begin{aligned} \langle P \rangle &= \frac{\int P dt}{t} \\ &= \frac{\int mat dt}{t} \\ &= \frac{mat}{2} \end{aligned}$$

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1.127. A small body of mass m is located on a horizontal plane at the point O . The body acquires a horizontal velocity v_0 . Find:

(a) the mean power developed by the friction force during the whole time of motion, if the friction coefficient $k = 0.27$, $m = 1.0$ kg, and $v_0 = 1.5$ m/s;

(b) the maximum instantaneous power developed by the friction force, if the friction coefficient varies as $k = \alpha x$, where α is a constant, and x is the distance from the point O .

A When it eventually stops in time t_0 :

$$v_0 = at_0$$

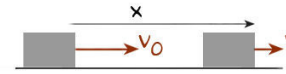
$$= kg t_0$$

$$\Rightarrow t_0 = \frac{v_0}{kg}$$

$$\Rightarrow \langle P \rangle = \frac{\Delta W}{\Delta t} = \frac{\Delta KE}{t_0} = \frac{-\frac{1}{2}mv_0^2}{v_0/kg}$$

$$= -\frac{mv_0 kg}{2} //$$

B



$$P = Fv = -(kmg)v$$

$$= -mg\alpha xv$$

$$= -mg\alpha x \sqrt{v_0^2 - g\alpha x^2}$$

Maximum when: $\frac{dP}{dx} = 0 \Rightarrow x = \frac{v_0}{\sqrt{2g\alpha}}$

$$\Rightarrow P_{\max} = -mg\alpha \frac{v_0}{\sqrt{2g\alpha}} \sqrt{v_0^2 - g\alpha \frac{v_0^2}{2g\alpha}} = -\frac{mv_0^2}{2\sqrt{2g\alpha}} //$$

$$W_{\text{ext}} = \Delta KE$$

$$W_{\text{friction}}$$

$$-\int_0^x kmg dx = \frac{1}{2}m(v^2 - v_0^2)$$

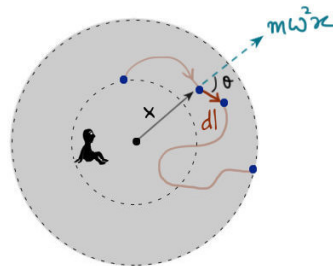
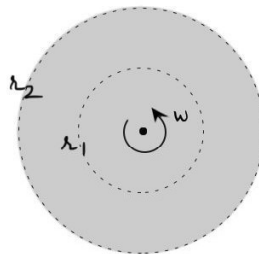
$$-\int_0^x \alpha x mg dx = \frac{1}{2}m(v^2 - v_0^2)$$

$$-\frac{mg\alpha x^2}{2} = \frac{1}{2}m(v^2 - v_0^2)$$

$$v = \sqrt{v_0^2 - g\alpha x^2}$$

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1.128. A small body of mass $m = 0.10$ kg moves in the reference frame rotating about a stationary axis with a constant angular velocity $\omega = 5.0$ rad/s. What work does the centrifugal force of inertia perform during the transfer of this body along an arbitrary path from point 1 to point 2 which are located at the distances $r_1 = 30$ cm and $r_2 = 50$ cm from the rotation axis?



$$dW = \vec{F} \cdot d\vec{l}$$

$$= (m\omega^2 r) dl \cos\theta$$

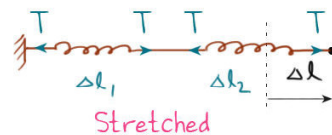
$$dx$$

$$\Rightarrow W = m\omega^2 \int_{r_1}^{r_2} x dx$$

$$= \frac{m\omega^2}{2} (r_2^2 - r_1^2) //$$

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1.129. A system consists of two springs connected in series and having the stiffness coefficients k_1 and k_2 . Find the minimum work to be performed in order to stretch this system by Δl .



Method 1

$$\begin{aligned}
 W_{\text{ext}} &= \Delta PE + \Delta KE \\
 &= \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 \\
 &= \frac{1}{2} k_1 \frac{k_2^2 \Delta l^2}{(k_1 + k_2)^2} + \frac{1}{2} k_2 \frac{k_1^2 \Delta l^2}{(k_1 + k_2)^2} \\
 &= \frac{1}{2} \frac{k_1 k_2 \Delta l^2}{(k_1 + k_2)} //
 \end{aligned}$$

In series spring combination

∴ Tension is same throughout,

$$k_1 \Delta l_1 = k_2 \Delta l_2$$

$$\text{Also, } \Delta l_1 + \Delta l_2 = \Delta l$$

$$\Delta l_1 = \frac{k_2}{k_1 + k_2} \Delta l, \quad \Delta l_2 = \frac{k_1}{k_1 + k_2} \Delta l$$

Method 2

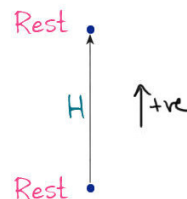
$$W_{\text{ext}} = \Delta PE = \frac{1}{2} k_{\text{eq}} \Delta l^2$$

$$k_{\text{eq}} = \frac{k_1 k_2}{k_1 + k_2}$$

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1.130. A body of mass m is hauled from the Earth's surface by applying a force $\vec{F}$ varying with the height of ascent y as $\vec{F} = 2(ay - 1)m\vec{g}$, where a is a positive constant. Find the work performed by this force and the increment of the body's potential energy in the gravitational field of the Earth over the first half of the ascent.

$$\vec{F} = 2(ay - 1)m\vec{g}$$



$$\begin{aligned}
 f_{\text{net}} &= \vec{F}_g + \vec{F} \\
 &= m\vec{g} + 2(ay - 1)m\vec{g} \\
 \Rightarrow m \frac{dv}{dy} &= -[-1 + 2ay]mg \\
 \Rightarrow \int_0^H v dv &= g \int_0^H (1 - 2ay) dy \\
 \Rightarrow 0 &= g \left(H - 2a \frac{H^2}{2} \right) \\
 \Rightarrow H &= 1/a
 \end{aligned}$$

$$\text{For half ascent: } y = \frac{H}{2} = \frac{1}{2a}$$

$$dW_F = \vec{F} \cdot d\vec{y} \quad \begin{matrix} \uparrow dy \\ \downarrow F \end{matrix}$$

$$\begin{aligned}
 \Rightarrow W_F &= - \int_0^{1/2a} 2(ay - 1)mg dy \\
 &= \frac{3mg}{4} //
 \end{aligned}$$

$$\Delta PE = mgH/2$$

$$= mg \frac{1}{2a} //$$

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1.131. The potential energy of a particle in a certain field has the form $U = \frac{a}{r^2} - \frac{b}{r}$, where a and b are positive constants, r is the distance from the centre of the field. **Find:**

(a) the value of r_0 corresponding to the equilibrium position of the particle; **examine** whether this position is **steady**;

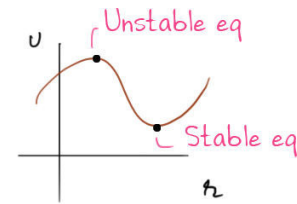
(b) the **maximum** magnitude of the **attraction force**; draw the plots $U(r)$ and $F_r(r)$ (the projections of the force on the radius vector $\mathbf{r}$).

A $U = \frac{a}{r^2} - \frac{b}{r}$

$$\frac{dU}{dr} = -\frac{2a}{r^3} + \frac{b}{r^2} \xrightarrow{\frac{dU}{dr}=0} r_0 = \frac{2a}{b}$$

$$\frac{d^2U}{dr^2} = \frac{6a}{r^4} - \frac{2b}{r^3} \xrightarrow{r_0 = \frac{2a}{b}} \frac{1}{8} \frac{b^4}{a^3} > 0 \Rightarrow \text{Stable equilibrium at } r = \frac{2a}{b}$$

Concept:



Stable/steady eq:

$$\frac{dU}{dr} = 0, \frac{d^2U}{dr^2} > 0$$

Unstable/unsteady eq:

$$\frac{dU}{dr} = 0, \frac{d^2U}{dr^2} < 0$$

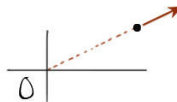
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Forces are central if $F = f(r) \hat{r}$

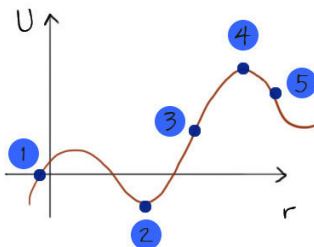
Central and attractive if $F = (-ve) \hat{r}$



Central and repulsive if $F = (+ve) \hat{r}$



$F = -ve \text{ Slope of } (U - r) \text{ graph} = -\frac{dU}{dr}$



- 1: Meaningless as PE is relative
- 2, 4: $F = 0$
- 2: Stable equilibrium
- 4: Unstable equilibrium
- 3: $F_{\max}$ (Attractive)
- 5: $F_{\max}$ (Repulsive)

B **maximum** magnitude of the **attraction force**:

$$U = \frac{a}{r^2} - \frac{b}{r}$$

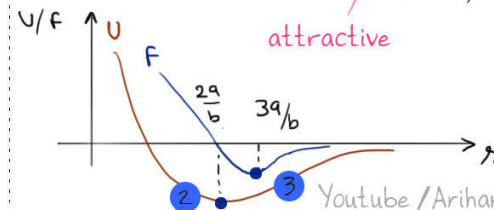
$$F = -\frac{dU}{dr} = \frac{2a}{r^3} - \frac{b}{r^2} (\hat{r})$$

$$F_{\max} \Rightarrow \frac{dF}{dr} = 0 \Rightarrow \frac{2a}{r^4} \left(-\frac{3}{4}\right) - \frac{b}{r^3} \left(-\frac{1}{1}\right) = 0$$

$$\Rightarrow r = \frac{3a}{b}, \infty$$

$$F_{\max} = F|_{3a/b} = -\frac{b^3}{27a^2}$$

attractive



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1.132. In a certain two-dimensional field of force the potential energy of a particle has the form $U = \alpha x^2 + \beta y^2$, where α and β are positive constants whose magnitudes are different. Find out:

- (a) whether this field is central;
 (b) what is the shape of the equipotential surfaces and also of the surfaces for which the magnitude of the vector of force $F = \text{const.}$

A Forces are central if $F = f(r) \hat{r}$ 1

Using, $F = -\frac{\partial U}{\partial x} \hat{i} - \frac{\partial U}{\partial y} \hat{j}$

$$U = \alpha x^2 + \beta y^2$$

$$F = -\frac{\partial U}{\partial x} \hat{i} - \frac{\partial U}{\partial y} \hat{j}$$

$$\Rightarrow F = -2\alpha x \hat{i} - 2\beta y \hat{j}$$

Which is not of the form of 1.

$\therefore$ Force is not central.

B At equipotential surfaces:
 $U = \text{constant}$

$$\Rightarrow \alpha x^2 + \beta y^2 = K$$

$$\Rightarrow \frac{x^2}{(K/\alpha)} + \frac{y^2}{(K/\beta)} = 1 \rightarrow \text{Ellipse}$$

Where Force is constant:

$$|-2\alpha x \hat{i} - 2\beta y \hat{j}| = K$$

$$\Rightarrow \frac{x^2}{K^2/4\alpha^2} + \frac{y^2}{K^2/4\beta^2} = 1 \rightarrow \text{Ellipse}$$

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1.133. There are two stationary fields of force $F = ay\hat{i}$ and $F = ax\hat{i} + by\hat{j}$, where $\hat{i}$ and $\hat{j}$ are the unit vectors of the x and y axes, and a and b are constants. Find out whether these fields are potential.

Can potential be defined, or
 Is the force field conservative?

We know that for conservative fields: $\vec{\nabla} \times \vec{F} = 0$

$$F = ay\hat{i}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & 0 & 0 \end{vmatrix} = -a\hat{k} \neq 0$$

$\therefore$ non conservative

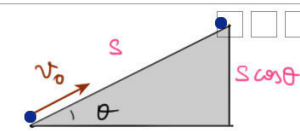
$$F = ax\hat{i} + by\hat{j}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax & by & 0 \end{vmatrix} = 0$$

$\therefore$ Conservative

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1.134. A body of mass m is pushed with the initial velocity v_0 up an inclined plane set at an angle α to the horizontal. The friction coefficient is equal to k . What distance will the body cover before it stops and what work do the friction forces perform over this distance?



$$a = -g \sin \theta - \mu g \cos \theta$$

$$v^2 = v_0^2 + 2as$$

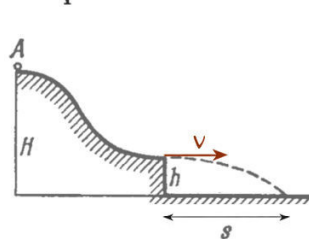
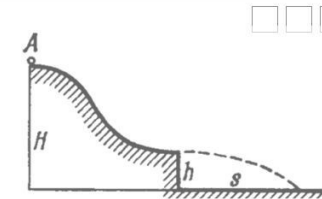
$$0 = v_0^2 - 2g(\sin \theta + \mu \cos \theta)s$$

$$\Rightarrow s = \frac{v_0^2}{2g(\sin \theta + k \cos \theta)}$$

$$W_{\text{friction}} = \vec{f}_k \cdot \vec{s} = -(\mu mg \cos \theta)s = -\frac{mv_0^2 k}{2(k + \tan \theta)}$$

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1.135. A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion (Fig. 1.30). What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?



$$v = \sqrt{2g(H-h)}$$

$$s = vt$$

$$= \sqrt{2g(H-h)} \sqrt{\frac{2h}{g}}$$

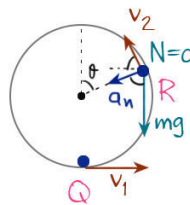
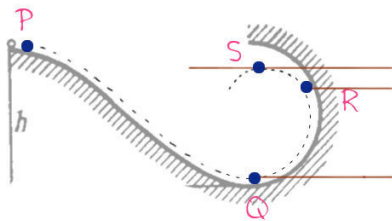
$$\Rightarrow s^2 = 4h(H-h)$$

Maximum at $h = H/2$

$$\Rightarrow s_{\text{max}}^2 = 4 \frac{H}{2} \left(H - \frac{H}{2}\right) \Rightarrow s_{\text{max}} = H$$

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1.136. A small body **A** starts sliding from the height h down an inclined groove passing into a half-circle of radius $h/2$ (Fig. 1.31). Assuming the friction to be negligible, find the velocity of the body at the highest point of its trajectory (after breaking off the groove).



$$\text{At Q: } mgh = \frac{1}{2}mv_1^2 \quad (1)$$

$$\text{At R: } mg \cos \theta = \frac{mv_2^2}{R} \quad (2) \quad \left\{ r_n = \frac{v_2^2}{R} \right\}$$

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 - mgR(1 + \cos \theta) \quad (3)$$

$$\text{From 1,2,3: } \cos \theta = 2/3, \quad v_2 = \sqrt{\frac{hg}{3}}$$

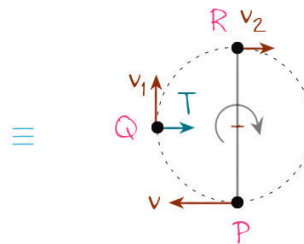
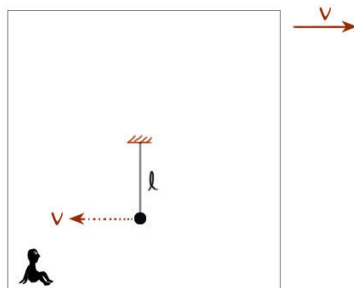
$$\therefore \text{At S: } v_s = v_2 \cos \theta = \frac{2}{3} \sqrt{\frac{hg}{3}}$$

- P to R Body sticks to the path
 R It breaks off and follows projectile path under gravity after that.
 S It reaches maximum height

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1.137. A ball of mass m is suspended by a thread of length l . With what minimum velocity has the point of suspension to be shifted in the horizontal direction for the ball to move along the circle about that point? What will be the tension of the thread at the moment it will be passing the horizontal position?

In order to complete the circle with minimum velocity, tension in the thread must not become zero until it reaches the top.



$$\text{At R: } F = ma \Rightarrow mg = \frac{mv_2^2}{l} \quad (1)$$

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv^2 - mg(2l) \quad (2)$$

$$\text{From 1,2: } v = \sqrt{5gl}$$

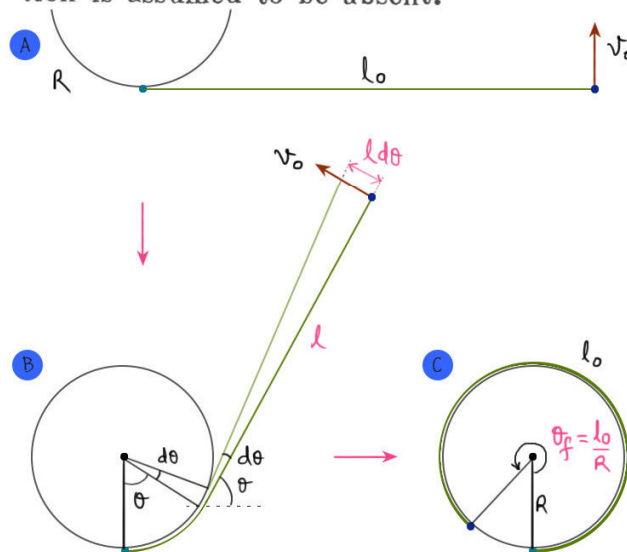
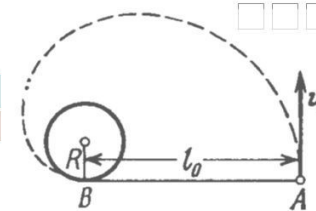
$$\text{At Q: } \frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 - mgl$$

$$T = \frac{mv_1^2}{l} = 3mg //$$

Relative velocity is v wrt frame.
 No pseudo force as acceleration of frame is zero.

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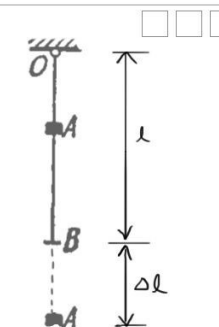
1.138. A horizontal plane supports a stationary vertical cylinder of radius R and a disc A attached to the cylinder by a horizontal thread AB of length l_0 (Fig. 1.32, top view). An initial velocity v_0 is imparted to the disc as shown in the figure. How long will it move along the plane until it strikes against the cylinder? The friction is assumed to be absent.



$$\begin{aligned}
 ds &= l d\theta \\
 &= (l_0 - R\theta) d\theta \\
 \Rightarrow S &= \int_0^{l_0/R} (l_0 - R\theta) d\theta \\
 &= l_0 \cdot \theta - \frac{R\theta^2}{2} \Big|_0^{l_0/R} \\
 &= \frac{l_0^2}{2R} \\
 \therefore t &= \frac{S}{v_0} = \frac{l_0^2}{2v_0 R} //
 \end{aligned}$$

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1.139. A smooth rubber cord of length l whose coefficient of elasticity is k is suspended by one end from the point O (Fig. 1.33). The other end is fitted with a catch B . A small sleeve A of mass m starts falling from the point O . Neglecting the masses of the thread and the catch, find the maximum elongation of the cord.



For maximum elongation, A must fall from point O .

By energy conservation:

(As mass comes to rest again at maximum elongation)

$$\begin{aligned}
 \Delta PE + \Delta KE &= 0 \\
 \Delta PE_g + \Delta PE_{\text{string}} &= 0 \\
 -mg(l + \Delta l) + \frac{1}{2}k\Delta l^2 &= 0
 \end{aligned}$$

$$\Rightarrow \Delta l = \frac{mg}{k} \left(1 + \sqrt{1 + \frac{2kl}{mg}} \right) //$$

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1.140. A small bar A resting on a smooth horizontal plane is attached by threads to a point P (Fig. 1.34) and, by means of a weightless pulley, to a weight B possessing the same mass as the bar itself. Besides, the bar is also attached to a point O by means of a light non-deformed spring of length $l_0 = 50$ cm and stiffness $k = 5 mg/l_0$, where m is the mass of the bar. The thread PA having been burned, the bar starts moving. Find its velocity at the moment when it is breaking off the plane.

When it breaks off, $N=0$

$$k \left[\frac{l_0}{\cos \theta} - l_0 \right] \cos \theta = mg$$

$$\Rightarrow \cos \theta = 1 - \frac{mg}{kl_0} \quad (1)$$

Energy conservation:

$$\Delta PE + \Delta KE = 0$$

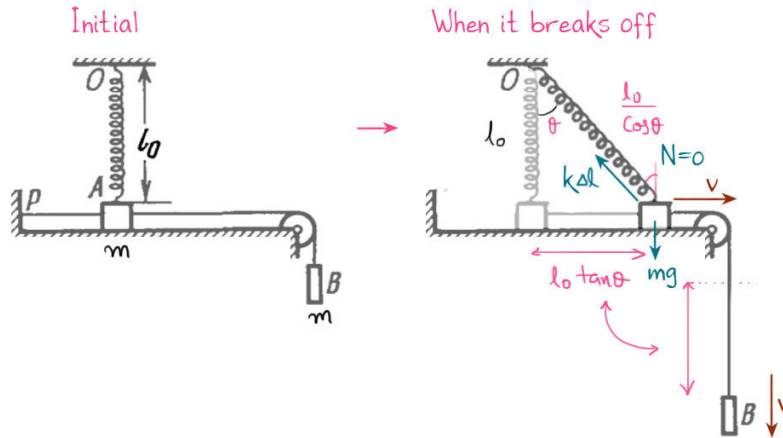
$$\Delta PE_{\text{Eq}} \quad \Delta PE_{\text{spring}} \quad \left[\frac{1}{2} (2m) v^2 - 0 \right]$$

$$-mg l_0 \tan \theta + \frac{1}{2} k \left[\frac{l_0}{\cos \theta} - l_0 \right]^2 + m v^2 = 0 \quad (2)$$

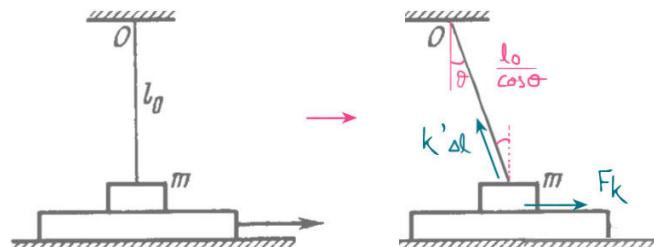
$$k = 5 \frac{mg}{l_0} \quad (3) \text{ (given in problem)}$$

$$\text{From 1, 2, 3: } v = \sqrt{\frac{19}{32} g l_0}$$

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1.141. A horizontal plane supports a plank with a bar of mass $m = 1.0$ kg placed on it and attached by a light elastic non-deformed cord of length $l_0 = 40$ cm to a point O (Fig. 1.35). The coefficient of friction between the bar and the plank equals $k = 0.20$. The plank is slowly shifted to the right until the bar starts sliding over it. It occurs at the moment when the cord deviates from the vertical by an angle $\theta = 30^\circ$. Find the work that has been performed by that moment by the friction force acting on the bar in the reference frame fixed to the plane.



$$\Delta l = \frac{l_0}{\cos \theta} - l_0 \quad (1)$$

$$k' \Delta l \sin \theta = k (mg - k' \Delta l \cos \theta) \quad (2)$$

Energy conservation:

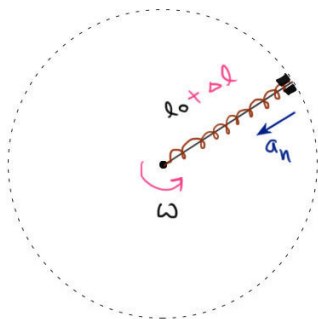
$$W_{\text{ext}} = (\Delta PE + \Delta KE)_{\text{system}}$$

$$W_{\text{friction}} = \frac{1}{2} k' \Delta l^2 = \frac{1}{2} (k' \Delta l) \Delta l$$

$$= \frac{1}{2} \left(\frac{kmg}{\sin \theta + k \cos \theta} \right) \cdot l_0 \left(\frac{1 - \cos \theta}{\cos \theta} \right)$$

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1.142. A smooth light horizontal rod AB can rotate about a vertical axis passing through its end A . The rod is fitted with a small sleeve of mass m attached to the end A by a weightless spring of length l_0 and stiffness k . What work must be performed to slowly get this system going and reaching the angular velocity ω ?



$$\Downarrow$$

$$v_n = 0$$

$$W_{ext} = (\Delta PE + \Delta KE)_{system}$$

$$\Rightarrow W_{ext} = \frac{1}{2} k \Delta l^2 + \frac{1}{2} m \omega^2 (l_0 + \Delta l)^2$$

$$= \frac{1}{2} \eta k l_0^2 \frac{(1+\eta)}{(1-\eta)^2}$$

$$F = ma$$

$$k \Delta l = m \omega^2 (l_0 + \Delta l)$$

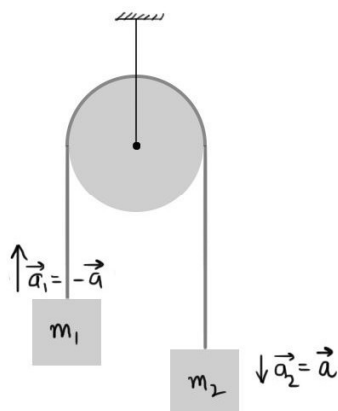
$$\Delta l = \left(\frac{m \omega^2}{k} \right) (l_0 + \Delta l)$$

Lets call it η

$$\Rightarrow \Delta l = \frac{\eta l_0}{1-\eta}$$

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1.143. A pulley fixed to the ceiling carries a thread with bodies of masses m_1 and m_2 attached to its ends. The masses of the pulley and the thread are negligible, friction is absent. Find the acceleration w_c of the centre of inertia of this system.



We know that,

$$\vec{a} = \frac{(m_2 - m_1)}{m_1 + m_2} \vec{g}$$

And,

$$a_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2} \left\{ \begin{array}{l} \text{differentiate} \\ \text{twice} \end{array} \right. x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\therefore \vec{a}_{cm} = \frac{m_1 (-\vec{a}) + m_2 \vec{a}}{m_1 + m_2}$$

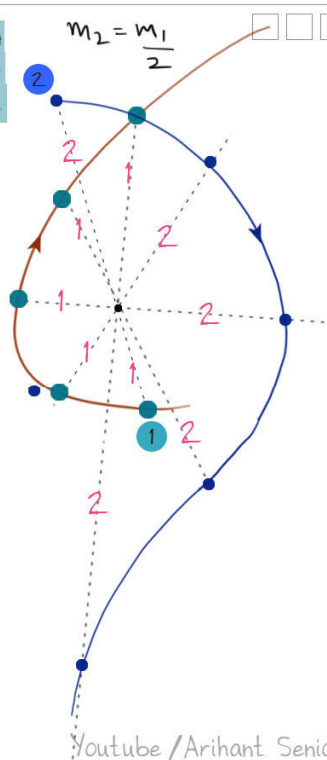
$$= \frac{(m_2 - m_1) \vec{a}}{m_1 + m_2} = \frac{(m_2 - m_1)^2}{(m_1 + m_2)^2} \vec{g}$$

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1.144. Two interacting particles form a closed system whose centre of inertia is at rest. Fig. 1.36 illustrates the positions of both particles at a certain moment and the trajectory of the particle of mass m_2 . Draw the trajectory of the particle of mass m_2 if $m_2 = m_1/2$.

Its how heavenly objects interact in space

Both will take sharp turns together



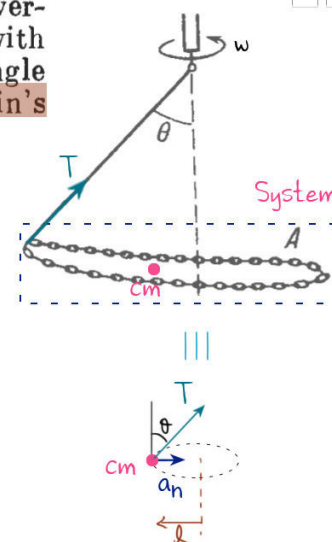
1.145. A closed chain A of mass $m = 0.36$ kg is attached to a vertical rotating shaft by means of a thread (Fig. 1.37), and rotates with a constant angular velocity $\omega = 35$ rad/s. The thread forms an angle $\theta = 45^\circ$ with the vertical. Find the distance between the chain's centre of gravity and the rotation axis, and the tension of the thread.

$$T \sin \theta = m \omega^2 l \quad (1)$$

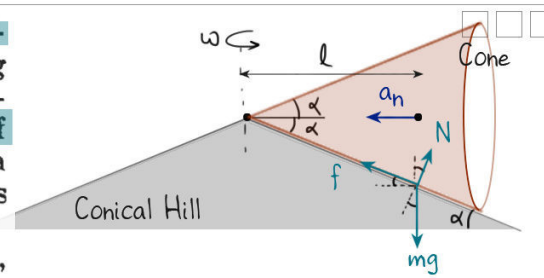
$$T \cos \theta = mg \quad (2)$$

From 1,2:

$$l = \frac{g \tan \theta}{\omega^2}, \quad T = \frac{mg}{\cos \theta}$$



1.146. A round cone A of mass $m = 3.2 \text{ kg}$ and half-angle $\alpha = 10^\circ$ rolls uniformly and without slipping along a round conical surface B so that its apex O remains stationary (Fig. 1.38). The centre of gravity of the cone A is at the same level as the point O and at a distance $l = 17 \text{ cm}$ from it. The cone's axis moves with angular velocity ω . Find:



(a) the static friction force acting on the cone A , if $\omega = 1.0 \text{ rad/s}$;

(b) at what values of ω the cone A will roll without sliding, if the coefficient of friction between the surfaces is equal to $k = 0.25$.

$$f \cos \alpha - N \sin \alpha = m \omega^2 l \quad (1)$$

$$f \sin \alpha + N \cos \alpha = mg \quad (2)$$

From 1,2

$$f = mg \sin \alpha + m \omega^2 l \cos \alpha$$

$$N = mg \cos \alpha - m \omega^2 l \sin \alpha$$

(B) At maximum ω to roll without slipping..

$$f < kN$$

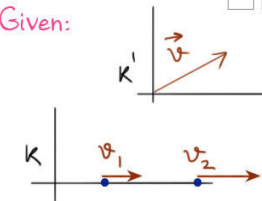
$$\therefore mg \sin \alpha + m \omega^2 l \cos \alpha > k(mg \cos \alpha - m \omega^2 l \sin \alpha)$$

$$\Rightarrow \omega \leq \sqrt{\frac{g(k - \tan \alpha)}{l(1 + k \tan \alpha)}}$$

Youtube / Arihant Senior

1.147. In the reference frame K two particles travel along the x axis, one of mass m_1 with velocity v_1 , and the other of mass m_2 with velocity v_2 . Find:

Given:



(a) the velocity V of the reference frame K' in which the cumulative kinetic energy of these particles is minimum;

(b) the cumulative kinetic energy of these particles in the K' frame.

$$A \quad KE|_{K'} = \frac{1}{2} m_1 v_{1|K'}^2 + \frac{1}{2} m_2 v_{2|K'}^2$$

$$= \frac{1}{2} m_1 (\bar{v}_1 - \bar{v})^2 + \frac{1}{2} m_2 (\bar{v}_2 - \bar{v})^2 \quad (1)$$

For minimum KE

$$\frac{d KE|_{K'}}{d \bar{v}} = 0 \Rightarrow \frac{1}{2} m_1 \cancel{2} (\bar{v}_1 - \bar{v}) (-1) + \frac{1}{2} m_2 \cancel{2} (\bar{v}_2 - \bar{v}) (-1) = 0$$

$$\Rightarrow \bar{v} = \frac{m_1 \bar{v}_1 + m_2 \bar{v}_2}{m_1 + m_2} \quad (2) \quad (\text{which is velocity of cm})$$

$\Rightarrow$ Kinetic energy of a system of particles is minimum in reference frame of cm (or traveling with same velocity as cm)

(B) Putting 2 in 1:

K.E wrt center of mass:

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\bar{v}_1 - \bar{v}_2|^2$$

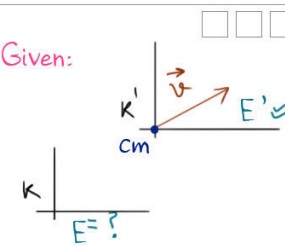
μ (reduced mass)

$$= \frac{1}{2} \mu |\bar{v}_1 - \bar{v}_2|^2$$

Youtube / Arihant Senior

1.148. The reference frame, in which the centre of inertia of a given system of particles is at rest, translates with a velocity $\vec{V}$ relative to an inertial reference frame K . The mass of the system of particles equals m , and the total energy of the system in the frame of the centre of inertia is equal to E' . Find the total energy E of this system of particles in the reference frame K .

Given:



$$E' = PE + KE'$$

$$= PE + \frac{1}{2} \sum m_i v_i'^2$$

$$E = PE + KE$$

$$= PE + \frac{1}{2} \sum m_i \left[\vec{v}_i|_{i|cm} + \vec{v}_{cm}|_K \right]^2$$

$$= PE + \frac{1}{2} \sum m_i (\vec{v}_i' + \vec{V})^2$$

$$= PE + \frac{1}{2} \sum m_i (\vec{v}_i' + \vec{V}) \cdot (\vec{v}_i' + \vec{V})$$

$$= PE + \underbrace{\frac{1}{2} \sum m_i v_i'^2}_{E'} + \underbrace{\frac{1}{2} \sum m_i V^2}_{\frac{1}{2} m V^2} + \underbrace{\sum m_i (\vec{v}_i' \cdot \vec{V})}_{= (\sum m_i \vec{v}_i') \cdot \vec{V} = 0}$$

momentum of system of particles wrt its center of mass = 0

$$= E' + \frac{1}{2} m V^2$$

Youtube / Arihant Senior

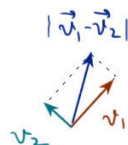
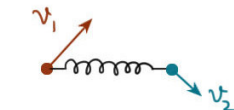
1.149. Two small discs of masses m_1 and m_2 interconnected by a weightless spring rest on a smooth horizontal plane. The discs are set in motion with initial velocities v_1 and v_2 whose directions are mutually perpendicular and lie in a horizontal plane. Find the total energy E of this system in the frame of the centre of inertia.

We know (proven in 1.147) that:

$$K.E \text{ wrt center of mass} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\vec{v}_1 - \vec{v}_2|^2 = \frac{1}{2} \mu |\vec{v}_1 - \vec{v}_2|^2$$

μ (reduced mass)

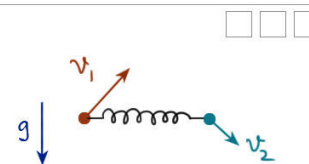
Sol:



$$\therefore KE_{cm} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$$

Youtube / Arihant Senior

1.150. A system consists of two small spheres of masses m_1 and m_2 interconnected by a weightless spring. At the moment $t = 0$ the spheres are set in motion with the initial velocities $\vec{v}_1$ and $\vec{v}_2$ after which the system starts moving in the Earth's uniform gravitational field. Neglecting the air drag, find the time dependence of the total momentum of this system in the process of motion and of the radius vector of its centre of inertia relative to the initial position of the centre.



A Method 1 $\vec{p} = \vec{p}_{\text{system}}|_{cm} + \vec{p}_{cm}|_{\text{ground}}$

$\downarrow$
 $M \vec{v}_{cm}$

B $(\vec{r}_f - \vec{r}_i)_{cm} = \vec{u}_{cm}t + \frac{1}{2}\vec{g}t^2$

$$= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}t + \frac{1}{2}\vec{g}t^2$$

$$= \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 //$$

Method 2

$$\vec{p}_0 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\vec{p}_t = m_1(\vec{v}_1 + \vec{g}t) + m_2(\vec{v}_2 + \vec{g}t)$$

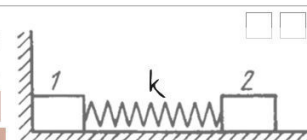
$$= (m_1\vec{v}_1 + m_2\vec{v}_2) + (m_1 + m_2)\vec{g}t \rightarrow \vec{p}_0 + M\vec{g}t //$$

$$= M(\vec{v}_{cmi} + \vec{g}t)$$

$$= M\vec{v}_{cmi} + M\vec{g}t$$

Youtube / Arihant Senior

1.151. Two bars of masses m_1 and m_2 connected by a weightless spring of stiffness k (Fig. 1.39) rest on a smooth horizontal plane. Bar 2 is shifted a small distance x to the left and then released. Find the velocity of the centre of inertia of the system after bar 1 breaks off the wall.



1 will break off as soon as spring starts to stretch.

At that moment, velocity of 2 is given by:

$$\frac{1}{2} kx^2 = \frac{1}{2} m_2 v_2^2 \Rightarrow v_2 = x \sqrt{\frac{k}{m_2}}$$

Spring force is internal.

So for the system after breaking off momentum is conserved.

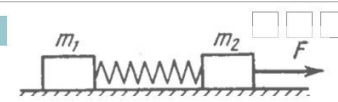
$$\therefore p_i = p_f$$

$$m_2 v_2 = (m_1 + m_2) v_{cm}$$

$$\Rightarrow v_{cm} = \frac{m_2 v_2}{m_1 + m_2} = \frac{m_2 x}{m_1 + m_2} \sqrt{\frac{k}{m_2}} //$$

Youtube / Arihant Senior

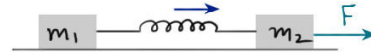
1.152. Two bars connected by a weightless spring of stiffness k and length (in the non-deformed state) l_0 rest on a horizontal plane. A constant horizontal force F starts acting on one of the bars as shown in Fig. 1.40. Find the maximum and minimum distances between the bars during the subsequent motion of the system, if the masses of the bars are:



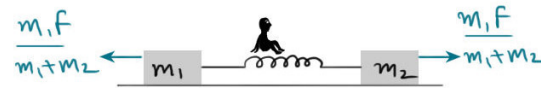
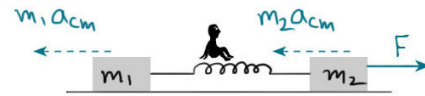
- (a) equal;
(b) equal to m_1 and m_2 , and the force F is applied to the bar of mass m_2 .

$$a_{cm} = \frac{F}{m_1 + m_2}$$

wrt Ground



wrt cm

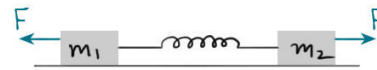


$$\Delta l_{max} = \frac{2F'}{k} = \frac{2}{k} \left(\frac{m_1 F}{m_1 + m_2} \right)$$

Youtube / Arihant Senior

Equal forces maximum extension case:

Initial: zero extension:



Final: maximum extension:

$$\Delta l = x_1 + x_2$$



Energy conservation:

$$F x_1 + F x_2 = \frac{1}{2} k \Delta l^2 + \cancel{\Delta KE}$$

$$\Rightarrow F \Delta l = \frac{1}{2} k \Delta l^2$$

$$\Rightarrow \Delta l = \frac{2F}{k} \text{ (Independent of masses)}$$

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As net force is zero, cm is at rest

$$m_1 x_1 = m_2 x_2 \quad (1)$$

Maximum extension:

$$\Delta l = x_1 + x_2 \quad (2)$$

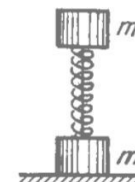
$$x_1 = \frac{m_2 \Delta l}{m_1 + m_2}, \quad x_2 = \frac{m_1 \Delta l}{m_1 + m_2}$$

Youtube / Arihant Senior

1.153. A system consists of two identical cubes, each of mass m , linked together by the compressed weightless spring of stiffness k (Fig. 1.41). The cubes are also connected by a thread which is burned through at a certain moment. Find:

(a) at what values of Δl , the initial compression of the spring, the lower cube will bounce up after the thread has been burned through:

(b) to what height h the centre of gravity of this system will rise if the initial compression of the spring $\Delta l = 7 mg/k$

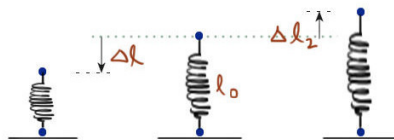


(For minimum Δl)

$$k \Delta l_2 = mg \quad (1)$$

$$\frac{1}{2} k \Delta l^2 = \frac{1}{2} k (\Delta l_2)^2 + mg(\Delta l + \Delta l_2) + \Delta k F \quad (2)$$

$$\text{From 1,2: } \Delta l = \frac{3mg}{k}, \quad -\frac{mg}{k}$$



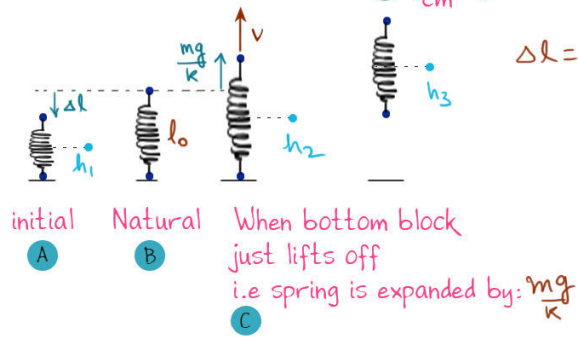
initial

Natural length

Elongation (when bottom block just lifts off)

Youtube / Arihant Senior

B • Position of cm



Upto liftoff:

$$\Delta l = \frac{7mg}{k} \quad (\text{given})$$

$$h_1 = \frac{l_0 - \Delta l}{2}$$

$$h_2 = \frac{l_0 + \frac{mg}{k}}{2}$$

$$h_2 - h_1 = \frac{4mg}{k} \quad (1)$$

$$\frac{1}{2}k\Delta l^2 = \frac{1}{2}mv^2 + \frac{1}{2}k\left(\frac{mg}{k}\right)^2 + mg\left(\Delta l + \frac{mg}{k}\right)$$

$$\Rightarrow v^2 = \frac{32mg^2}{k} \Rightarrow v_{cm}^2 = \frac{v^2}{4} = \frac{8mg^2}{k}$$

After liftoff to maximum height:

$$h_3 - h_2 = \frac{v_{cm}^2}{2g} = \frac{4mg}{k} \quad (2)$$

$$\text{From 1,2 } h_3 - h_1 = (h_3 - h_2) + (h_2 - h_1) = \frac{8mg}{k}$$

Youtube / Arihant Senior

1.154. Two identical buggies 1 and 2 with one man in each move without friction due to inertia along the parallel rails toward each other. When the buggies get opposite each other, the men exchange their places by jumping in the direction perpendicular to the motion direction. As a consequence, buggy 1 stops and buggy 2 keeps moving in the same direction, with its velocity becoming equal to v . Find the initial velocities of the buggies $\vec{v}_1$ and $\vec{v}_2$ if the mass of each buggy (without a man) equals M and the mass of each man m .

Momentum conservation for both the systems:

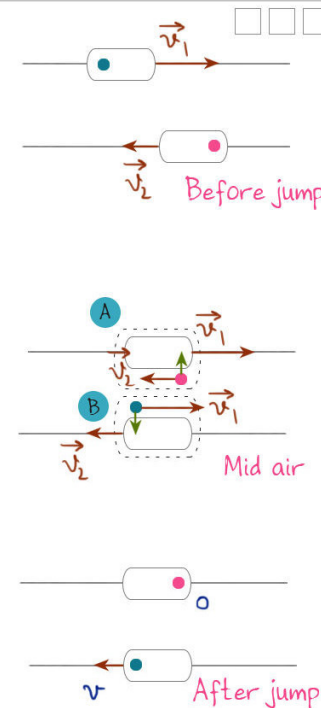
$$p_i = p_f$$

$$M\vec{v}_1 + m\vec{v}_2 = 0 \quad (1)$$

$$M\vec{v}_2 + m\vec{v}_1 = (M+m)\vec{v} \quad (2)$$

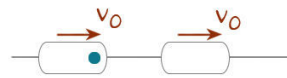
From 1,2:

$$\vec{v}_1 = \frac{-m\vec{v}}{M-m}, \quad \vec{v}_2 = \frac{M\vec{v}}{M-m}$$



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1.155. Two identical buggies move one after the other due to inertia (without friction) with the same velocity v_0 . A man of mass m rides the rear buggy. At a certain moment the man jumps into the front buggy with a velocity u relative to his buggy. Knowing that the mass of each buggy is equal to M , find the velocities with which the buggies will move after that.

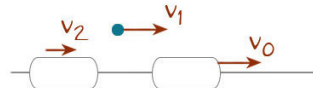


Buggy 1:

$$(m+M)\bar{v}_0 = m\bar{v}_1 + M\bar{v}_2$$

$$\bar{v}_1 - \bar{v}_2 = \bar{u}$$

$$v_2 = v_0 - \frac{mu}{m+M}$$



System:

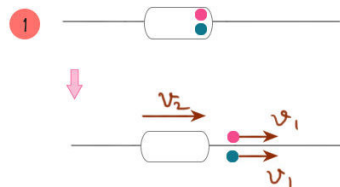
$$(2M+m)v_0 = Mv_2 + (m+M)v_3$$

$$\Rightarrow v_3 = v_0 + \frac{mMu}{(m+M)^2}$$



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1.156. Two men, each of mass m , stand on the edge of a stationary buggy of mass M . Assuming the friction to be negligible, find the velocity of the buggy after both men jump off with the same horizontal velocity u relative to the buggy: (1) simultaneously; (2) one after the other. In what case will the velocity of the buggy be greater and how many times?

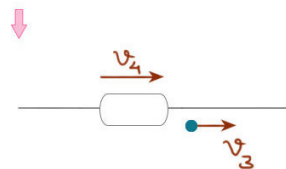
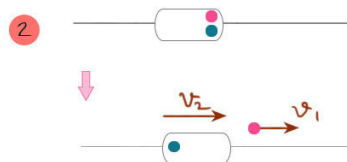


$$M\bar{v}_2 + 2m\bar{v}_1 = 0$$

$$\bar{v}_1 - \bar{v}_2 = \bar{u}$$

solving ↓

$$v_2 = \frac{-2mu}{2m+M}$$



After first jump:

$$(M+m)\bar{v}_2 + m\bar{v}_1 = 0$$

$$\bar{v}_1 - \bar{v}_2 = \bar{u}$$

solving ↓

$$v_2 = \frac{-mu}{M+2m}$$

After second jump:

$$(M+m)\bar{v}_2 = M\bar{v}_4 + m\bar{v}_3$$

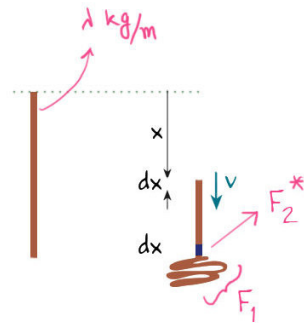
$$\bar{v}_3 - \bar{v}_4 = \bar{u}$$

solving ↓

$$v_4 = \frac{-m(3m+2M)u}{(m+M)(2m+M)}$$

Youtube / Arihant Senior

1.157. A chain hangs on a thread and touches the surface of a table by its lower end. Show that after the thread has been burned through, the force exerted on the table by the falling part of the chain at any moment is twice as great as the force of pressure exerted by the part already resting on the table.



Let the force on table due to chain already on floor be F_1 and, Force on table due to chain hitting the surface be F_2^*

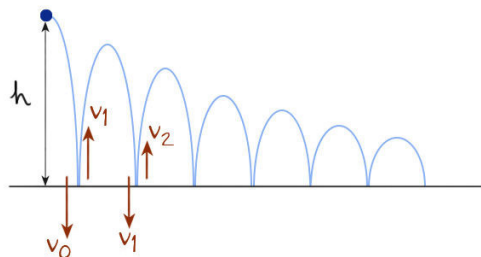
$$F_1 = mg = \lambda x g //$$

$$F_2^* = \left| \frac{dp}{dt} \right| = \frac{dm(v-0)}{dt} = v \frac{dm}{dt} = v \lambda \frac{dx}{dt} = \lambda v^2 = 2\lambda g x //$$

$$\therefore F_2^* = 2 F_1 //$$

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1.158. A steel ball of mass $m = 50 \text{ g}$ falls from the height $h = 1.0 \text{ m}$ on the horizontal surface of a massive slab. Find the cumulative momentum that the ball imparts to the slab after numerous bounces, if every impact decreases the velocity of the ball $\eta = 1.25$ times.



$$v_0 = \sqrt{2gh} \quad (1)$$

$$\left. \begin{aligned} v_1 &= \frac{v_0}{\eta} \\ v_2 &= \frac{v_1}{\eta} = \frac{v_0}{\eta^2} \\ v_3 &= \frac{v_2}{\eta} = \frac{v_0}{\eta^3} \\ &\vdots \end{aligned} \right\} (2)$$

Momentum imparted by ball after:

$$1^{\text{st}} \text{ bounce: } m(\vec{v}_0 - \vec{v}_1) = m(v_0 + v_1)$$

$$2^{\text{nd}} \text{ bounce: } = m(v_1 + v_2)$$

$\vdots$

$$\text{Total: } \vec{p} = m(v_0 + 2v_1 + 2v_2 + \dots)$$

$$\text{From 2: } = m \left[v_0 + 2 \frac{v_0}{\eta} \left(1 + \frac{1}{\eta} + \frac{1}{\eta^2} + \dots \right) \right]$$

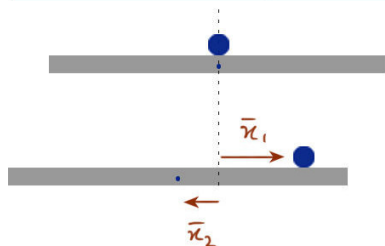
$$= m v_0 \left[1 + \frac{2}{1 - \frac{1}{\eta}} \right]$$

$$\text{From 1: } = m v_0 \frac{(\eta + 1)}{(\eta - 1)} = m \sqrt{2gh} \frac{\eta + 1}{\eta - 1}$$

Youtube / Arihant Senior

1.159. A raft of mass M with a man of mass m aboard stays motionless on the surface of a lake. The man moves a distance l' relative to the raft with velocity $v'(t)$ and then stops. Assuming the water resistance to be negligible, find:

- the displacement of the raft l relative to the shore;
- the horizontal component of the force with which the man acted on the raft during the motion.



A Given,
 $\vec{x}_1 - \vec{x}_2 = l'$ (1)

cm is at rest:

$\therefore m\vec{x}_1 + M\vec{x}_2 = 0$ (2)

Solving

$\vec{x}_2 = -\frac{m}{m+M}l'$

diff..

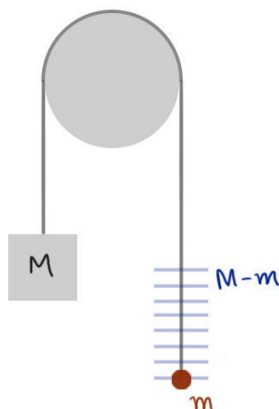
$\vec{v}_2 = -\frac{m}{m+M}v'$

$\Rightarrow F = -\frac{mM}{(m+M)}\frac{dv'}{dt}$

B Force on raft = $M\frac{dv_2}{dt}$

Youtube / Arihant Senior

1.160. A stationary pulley carries a rope whose one end supports a ladder with a man and the other end the counterweight of mass M . The man of mass m climbs up a distance l' with respect to the ladder and then stops. Neglecting the mass of the rope and the friction in the pulley axle, find the displacement l of the centre of inertia of this system.



Let the displacements be:

m

$\vec{x}_1$

$M-m$

$\vec{x}_2$

M

$-\vec{x}_2$

Given:

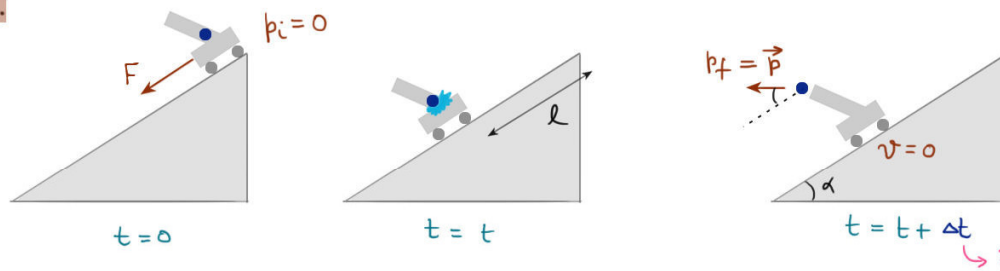
$\vec{x}_1 - \vec{x}_2 = l'$

$\therefore$ Displacement of cm: $= \frac{m\vec{x}_1 + (M-m)\vec{x}_2 + M(-\vec{x}_2)}{2M}$

$= \frac{m}{2M}(\vec{x}_1 - \vec{x}_2) = \frac{ml'}{2M}$

Youtube / Arihant Senior

1.161. A cannon of mass M starts sliding freely down a smooth inclined plane at an angle α to the horizontal. After the cannon covered the distance l , a shot was fired, the shell leaving the cannon in the horizontal direction with a momentum p . As a consequence, the cannon stopped. Assuming the mass of the shell to be negligible, as compared to that of the cannon, determine the duration of the shot.



For the system (therefore neglecting internal forces) parallel to plane:

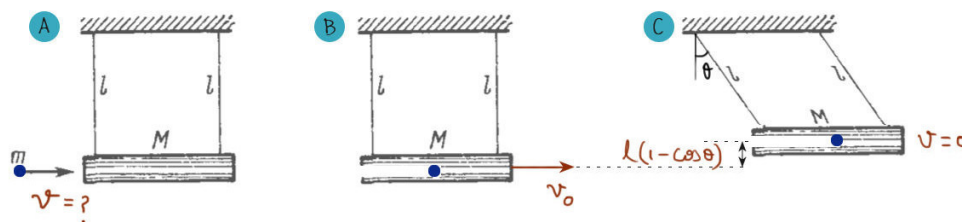
Impulse = change in momentum

$$\begin{aligned} \therefore F(t + \Delta t) &= p \cos \alpha \Rightarrow Mg \sin \alpha \left(\sqrt{\frac{2l}{g \sin \alpha}} + \Delta t \right) = p \cos \alpha \Rightarrow \Delta t = \frac{p \cos \alpha - M \sqrt{2g \sin \alpha l}}{Mg \sin \alpha} // \\ \text{Mg sin } \alpha \quad l &= \frac{1}{2} (g \sin \alpha) t^2 \end{aligned}$$

Youtube / Arihant Senior

1.162. A horizontally flying bullet of mass m gets stuck in a body of mass M suspended by two identical threads of length l (Fig. 1.42). As a result, the threads swerve through an angle θ . Assuming $m \ll M$, find:

- (a) the velocity of the bullet before striking the body;
(b) the fraction of the bullet's initial kinetic energy that turned into heat.



A-B: Momentum conservation

$$mv = (m+M)v_0 \quad (1)$$

B-C: Energy conservation

$$\frac{1}{2}(m+M)v_0^2 = (m+M)gl(1-\cos\theta) \quad (2)$$

From 1,2:

$$v = \frac{2(M+m)}{m} \sqrt{2gl(1-\cos\theta)} //$$

$$\text{Heat} = \frac{\kappa E_i - \kappa E_f}{\kappa E_i}$$

$$= 1 - \frac{\kappa E_f}{\kappa E_i}$$

$$= 1 - \frac{\frac{1}{2}(m+M)v_0^2}{\frac{1}{2}mv^2}$$

$$\frac{v_0}{v} \text{ from 1}$$

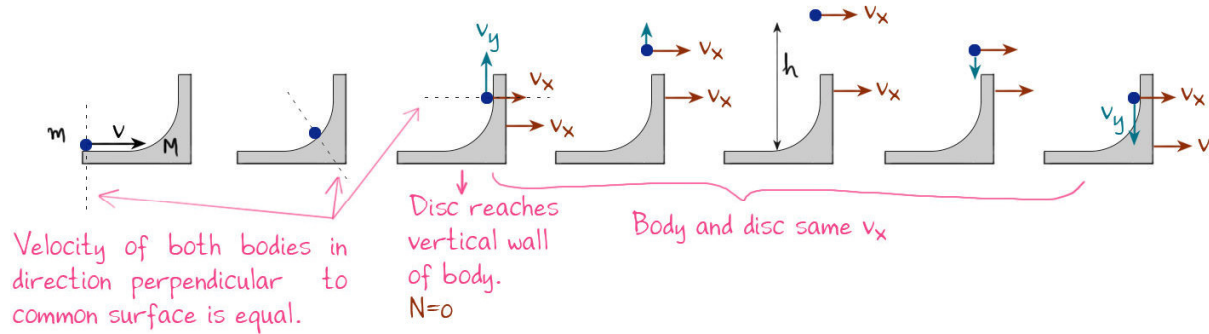
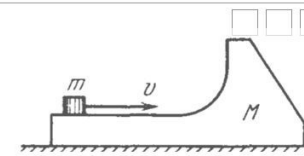
$$= 1 - \frac{(M+m) \cdot m^2}{m^2 (M+m)^2}$$

$$= 1 - \frac{m}{M+m} \rightarrow \approx 0$$

$$\approx 1 - m/M //$$

Youtube / Arihant Senior

1.163. A body of mass M (Fig. 1.43) with a small disc of mass m placed on it rests on a smooth horizontal plane. The disc is set in motion in the horizontal direction with velocity v . To what height (relative to the initial level) will the disc rise after breaking off the body M ? The friction is assumed to be absent.



Momentum conservation in x direction:

$$mv = (m+M)v_x \quad (1)$$

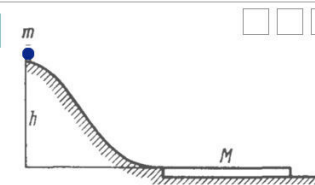
Energy conservation:

$$\frac{1}{2}mv^2 = \frac{1}{2}(M+m)v_x^2 + mgh \quad (2)$$

From 1,2: $h = \frac{Mv^2}{2g(M+m)}$

Youtube / Arihant Senior

1.164. A small disc of mass m slides down a smooth hill of height h without initial velocity and gets onto a plank of mass M lying on the horizontal plane at the base of the hill (Fig. 1.44). Due to friction between the disc and the plank the disc slows down and, beginning with a certain moment, moves in one piece with the plank.



(1) Find the total work performed by the friction forces in this process.

(2) Can it be stated that the result obtained does not depend on the choice of the reference frame?

① Friction is internal force. So conserving momentum in X direction for disc plank system:

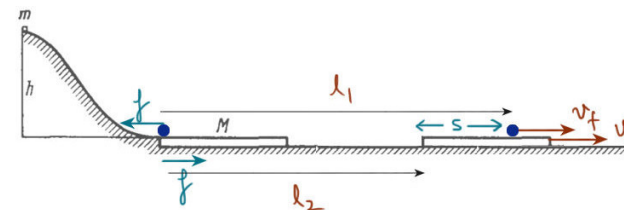
$$m\sqrt{2gh} = (m+M)v_f \Rightarrow v_f = \frac{m\sqrt{2gh}}{m+M}$$

Energy conservation:

$$mgh + W_{f_2} = \frac{1}{2}(m+M)\frac{m^2(2gh)}{(m+M)^2}$$

$$\Rightarrow W_{f_2} = -mgh\left(\frac{M}{m+M}\right)$$

②



$$\text{Total } W_f: -f l_1 + f l_2 = -f(l_1 - l_2) = -f s$$

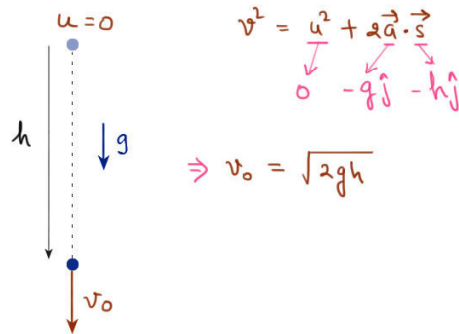
Same in all reference frames

Youtube / Arihant Senior

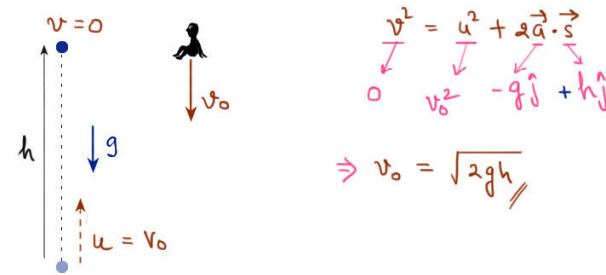
1.165. A stone falls down without initial velocity from a height h onto the Earth's surface. The air drag assumed to be negligible, the stone hits the ground with velocity $v_0 = \sqrt{2gh}$ relative to the Earth. Obtain the same formula in terms of the reference frame "falling" to the Earth with a constant velocity v_0 .

Method 1

A w.r.t ground



B w.r.t frame



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Method 2

A w.r.t ground

Impulse: $\vec{J} = \vec{F}t = -m\sqrt{2gh}\hat{j}$

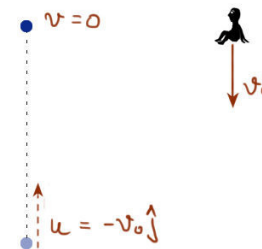
$-mg\hat{j} \cdot \frac{\sqrt{2h}}{g}$

$\Rightarrow -m\sqrt{2gh}\hat{j} = m(\vec{v}_0 - 0)$

$\Rightarrow \vec{v}_0 = -\sqrt{2gh}\hat{j}$

$\Rightarrow |\vec{v}_0| = \sqrt{2gh}$

B w.r.t frame



Impulse is same as was in ground frame as:

- 1 Time passes equally in all reference frames
- 2 F is same in inertial reference frames.

$\therefore \vec{J} = -m\sqrt{2gh}\hat{j} = m(\vec{v}_0 - (-v_0\hat{j}))$

$\Rightarrow v_0 = \sqrt{2gh}$

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1.166. A particle of mass 1.0 g moving with velocity $\vec{v}_1 = 3.0\hat{i} - 2.0\hat{j}$ experiences a perfectly inelastic collision with another particle of mass 2.0 g and velocity $\vec{v}_2 = 4.0\hat{j} - 6.0\hat{k}$. Find the velocity of the formed particle (both the vector $\vec{v}$ and its modulus), if the components of the vectors $\vec{v}_1$ and $\vec{v}_2$ are given in the SI units.

Conserving momentum on system, as forces are internal

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$

$$\Rightarrow \vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} //$$

(Same as velocity of center of mass)

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1.167. Find the increment of the kinetic energy of the closed system comprising two spheres of masses m_1 and m_2 due to their perfectly inelastic collision, if the initial velocities of the spheres were equal to $\vec{v}_1$ and $\vec{v}_2$.

From momentum conservation:

$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$\Delta KE = \frac{1}{2}(m_1 + m_2)\vec{v} \cdot \vec{v} - \left(\frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 \right)$$

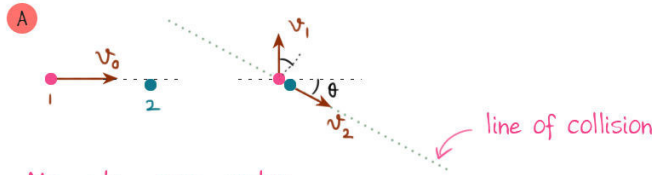
$$= \frac{1}{2}(m_1 + m_2) \frac{(m_1^2\vec{v}_1^2 + m_2^2\vec{v}_2^2 + 2m_1m_2\vec{v}_1 \cdot \vec{v}_2)}{(m_1 + m_2)^2} - \left(\frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 \right)$$

$$= -\frac{1}{2} \frac{m_1m_2}{m_1 + m_2} (\vec{v}_1^2 + \vec{v}_2^2 - \underbrace{2\vec{v}_1 \cdot \vec{v}_2}_{2v_1v_2\cos\theta}) = -\frac{1}{2} \frac{m_1m_2}{(m_1 + m_2)} |\vec{v}_1 - \vec{v}_2|^2 = -\frac{1}{2} \mu |\vec{v}_1 - \vec{v}_2|^2 //$$

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1.168. A particle of mass m_1 experienced a perfectly elastic collision with a stationary particle of mass m_2 . What fraction of the kinetic energy does the striking particle lose, if

- (a) it recoils at right angles to its original motion direction;
 (b) the collision is a head-on one?



Momentum conservation:

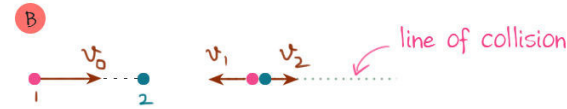
(in X direction) $m_1 v_0 = m_2 v_2 \cos \theta$ ①

(in Y direction) $m_1 v_1 = m_2 v_2 \sin \theta$ ②

v of approach = v of separation (along line of collision)

$v_0 \cos \theta = v_1 \sin \theta + v_2$ ③

$\frac{KE_{lost}}{KE_i} = \frac{\frac{1}{2} m_1 v_2^2}{\frac{1}{2} m_1 v_0^2} = \frac{v_2^2}{v_0^2} = \frac{2m_1}{m_1 + m_2}$ //



Momentum conservation:

$m_1 v_0 = m_2 v_2 - m_1 v_1$ ①

v of approach = v of separation

$v_0 = v_1 + v_2$ ②

$\frac{KE_{lost}}{KE_i} = \frac{v_2^2}{v_0^2} = \frac{4m_1 m_2}{(m_1 + m_2)^2}$ //

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1.169. Particle 1 experiences a perfectly elastic collision with a stationary particle 2. Determine their mass ratio, if

- (a) after a head-on collision the particles fly apart in the opposite directions with equal velocities;
 (b) the particles fly apart symmetrically relative to the initial motion direction of particle 1 with the angle of divergence $\theta = 60^\circ$.

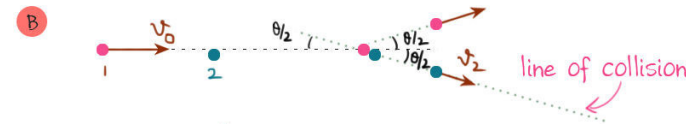


$m_1 v_0 = (m_2 - m_1) v$ ①

$v_0 = 2v$ ②

$\frac{1}{2} \Rightarrow m_1 = \frac{m_2 - m_1}{2}$

$\Rightarrow \frac{m_1}{m_2} = \frac{1}{3}$ //



p_x $m_1 v_0 = (m_1 v_1 + m_2 v_2) \cos \theta/2$ ①

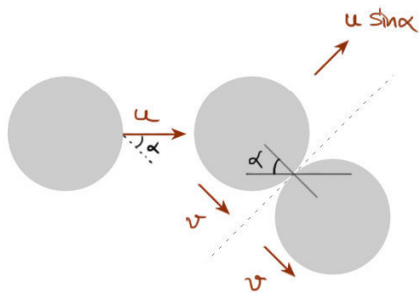
p_y $m_1 v_1 \sin \theta/2 = m_2 v_2 \sin \theta/2$ ②

$v_s = v_a$ $v_0 \cos \theta/2 = v_2 - v_1 \cos \theta$ ③

$\frac{1}{2}$ & Dividing numerator and denominator by $m_2 v_1 \Rightarrow \frac{m_1}{m_2} = 1 + 2 \cos \theta$ //

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1.170. A ball moving translationally collides elastically with another, stationary, ball of the same mass. At the moment of impact the angle between the straight line passing through the centres of the balls and the direction of the initial motion of the striking ball is equal to $\alpha = 45^\circ$. Assuming the balls to be smooth, find the fraction η of the kinetic energy of the striking ball that turned into potential energy at the moment of the maximum deformation.



At maximum deformation velocities of both balls along line of collision will be equal

CLM along line of collision:

$$mu \cos \alpha = 2mv \quad (1)$$

$$\begin{aligned} \frac{PE}{KE_i} &= \frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i} \\ &= 1 - \frac{\frac{1}{2}mu^2 \sin^2 \alpha + \frac{1}{2}(2m)v^2}{\frac{1}{2}mu^2} \quad (2) \\ &= \frac{\cos^2 \alpha}{2} \end{aligned}$$

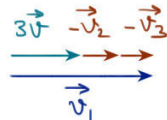
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1.171. A shell flying with velocity $v = 500$ m/s bursts into three identical fragments so that the kinetic energy of the system increases $\eta = 1.5$ times. What maximum velocity can one of the fragments obtain?

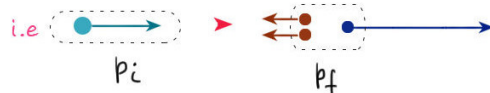
Lets maximise v_1 .

A CLM: $3\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ (To decide direction)

For a given magnitude of v , v_2 and v_3 , this arrangement gives maximum value of v_1 .



$$\therefore v_1 = 3v + v_2 + v_3 \quad (1)$$



(To decide magnitude)

B Given: $3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \quad (2)$
Maximum $\Rightarrow$ Minimum

$$AM > GM \Rightarrow \frac{v_2^2 + v_3^2}{2} \geq \sqrt{v_2^2 v_3^2}$$

$$\therefore \text{For minimum LHS} \Rightarrow \left(\frac{v_2^2 + v_3^2}{2} \right) = \sqrt{v_2^2 v_3^2}$$

$$\Rightarrow (v_2 - v_3)^2 = 0$$

$$\Rightarrow v_2 = v_3 \quad (3)$$

From 1,2,3:

$$v_1 = (1 + \sqrt{2(\eta - 1)})v$$

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1.172. Particle 1 moving with velocity $v = 10 \text{ m/s}$ experienced a head-on collision with a stationary particle 2 of the same mass. As a result of the collision, the kinetic energy of the system decreased by $\eta = 1.0\%$. Find the magnitude and direction of the velocity of particle 1 after the collision.



CLM: $mv = mv_1 + mv_2$ (1)

Given: $\eta = \frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i}$

$$= 1 - \frac{\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2}{\frac{1}{2}mv^2}$$

$$\Rightarrow \eta = 1 - \frac{v_1^2 + v_2^2}{v^2} \quad (2)$$

From 1,2:

$$v_1 = \frac{v}{2} (1 \pm \sqrt{1-2\eta}) //$$

root with + sign gives v_2
root with - sign gives v_1 .

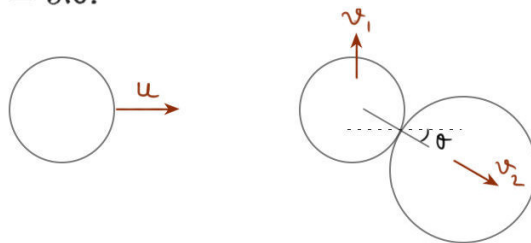
$\therefore$ for $\eta < 1$

$$v_1 = \frac{v}{2} (1 - (1 - 2\eta))$$

$$= \frac{\eta v}{2} //$$

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1.173. A particle of mass m having collided with a stationary particle of mass M deviated by an angle $\pi/2$ whereas the particle M recoiled at an angle $\theta = 30^\circ$ to the direction of the initial motion of the particle m . How much (in per cent) and in what way has the kinetic energy of this system changed after the collision, if $M/m = 5.0$?



CLM in X: $mu = Mv_2 \cos \theta$ (1)

CLM in Y: $m v_1 = M v_2 \sin \theta$ (2)

Percent change in KE:

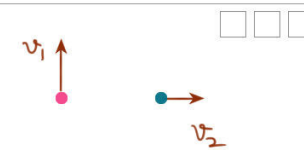
$$\frac{\Delta KE}{KE_i} \times 100 = \frac{\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 - \frac{1}{2}mu^2}{\frac{1}{2}mu^2} \times 100$$

From 1,2:

$$= \left(\tan^2 \theta + \frac{m}{M} \sec^2 \theta - 1 \right) 100 //$$

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1.174. A closed system consists of two particles of masses m_1 and m_2 which move at right angles to each other with velocities v_1 and v_2 . Find:



- (a) the momentum of each particle and
(b) the total kinetic energy of the two particles in the reference frame fixed to their centre of inertia.

A We know, $v_{cm} = \frac{m_1 v_1 \hat{j} + m_2 v_2 \hat{i}}{m_1 + m_2}$

$$\therefore v_{1,cm} = v_1 - v_{cm} = v_1 \hat{j} - \frac{m_1 v_1 \hat{j} + m_2 v_2 \hat{i}}{m_1 + m_2}$$

$$= \frac{m_2 (v_2 \hat{i} - v_1 \hat{j})}{m_1 + m_2}$$

$$\Rightarrow \vec{p}_{1,cm} = m_1 v_{1,cm} = \frac{m_1 m_2 (v_2 \hat{i} - v_1 \hat{j})}{m_1 + m_2} //$$

$$\vec{p}_{1,cm} + \vec{p}_{2,cm} = 0 \Rightarrow \vec{p}_{2,cm} = \frac{-m_1 m_2 (v_2 \hat{i} - v_1 \hat{j})}{m_1 + m_2} //$$

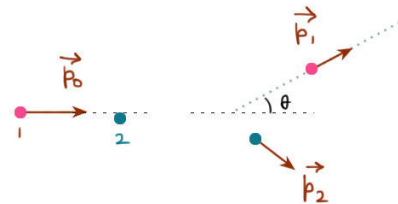
B $KE_{cm} = \frac{1}{2} m_1 v_{1,cm}^2 + \frac{1}{2} m_2 v_{2,cm}^2$

$$= \frac{1}{2} m_1 \frac{m_2^2 (v_2^2 + v_1^2)}{(m_1 + m_2)^2} + \frac{1}{2} m_2 \frac{m_1^2 (v_2^2 + v_1^2)}{(m_1 + m_2)^2}$$

$$= \frac{1}{2} \frac{m_1 m_2 (v_1^2 + v_2^2)}{m_1 + m_2} //$$

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1.175. A particle of mass m_1 collides elastically with a stationary particle of mass m_2 ($m_1 > m_2$). Find the maximum angle through which the striking particle may deviate as a result of the collision.



conserving momentum

$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2$$

$$\Rightarrow |\vec{p}_0 - \vec{p}_1| = |\vec{p}_2|$$

$$\Rightarrow p_0^2 + p_1^2 - 2p_0 p_1 \cos \theta = p_2^2 \quad (1)$$

conserving energy

$$\frac{p_0^2}{2m_1} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \quad (2)$$

Eliminating p_2 from 1,2:

$$\left(1 + \frac{m_2}{m_1}\right) p_1^2 - 2p_0 \cos \theta p_1 + \left(1 - \frac{m_2}{m_1}\right) p_0^2 = 0$$

For a given p_0 , for p_1 to have real roots:

$$D \geq 0 \Rightarrow \cos^2 \theta \geq \left(1 - \frac{m_2^2}{m_1^2}\right)$$

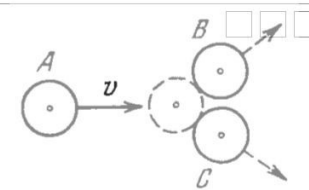
$$\Rightarrow \sin^2 \theta \leq \frac{m_2^2}{m_1^2} \quad \left\{ \begin{array}{l} \because 0 < \theta < \pi \\ \sin \theta \text{ can't be -ve} \end{array} \right\}$$

$$\therefore \sin \theta \leq \frac{m_2}{m_1}$$

$$\Rightarrow \sin \theta_{\max} = \frac{m_2}{m_1} //$$

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1.176. Three identical discs A, B, and C (Fig. 1.45) rest on a smooth horizontal plane. The disc A is set in motion with velocity $\mathbf{v}$ after which it experiences an elastic collision simultaneously with the discs B and C. The distance between the centres of the latter discs prior to the collision is η times greater than the diameter of each disc.



Find the velocity of the disc A after the collision. At what value of η will the disc A recoil after the collision; stop; move on?

Given: $\sin \theta = \frac{\frac{\eta d}{2}}{d} = \frac{\eta}{2}$ (1)

Conserving momentum: $m\mathbf{v} = m\mathbf{v}_1 + 2m\mathbf{v}_2 \cos \theta$ (2)

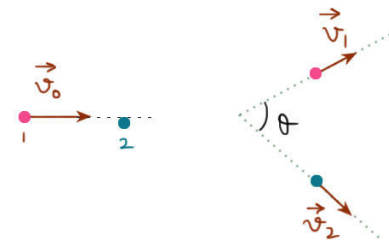
Velocity of Approach = VoS: $v \cos \theta = v_2 - v_1 \cos \theta$ (3)

From 1,2,3: $v_1 = \frac{v(\eta^2 - 2)}{6 - \eta^2}$

$\therefore$ Disc will Continue if: $\eta > \sqrt{2}$ $\rightarrow$ (including large values of η as ball will continue without collision)
 Recoil if: $\eta < \sqrt{2}$
 Stop if: $\eta = \sqrt{2}$

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1.177. A molecule collides with another, stationary, molecule of the same mass. Demonstrate that the angle of divergence (a) equals 90° when the collision is ideally elastic; (b) differs from 90° when the collision is inelastic.



For elastic collision:

conserving momentum

$$\vec{v}_0 = \vec{v}_1 + \vec{v}_2 \Rightarrow v_0^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos \theta \quad (1)$$

conserving energy

$$v_0^2 = v_1^2 + v_2^2 \quad (2)$$

$\therefore$ From 1,2: $\cos \theta = 0 \Rightarrow \theta = \pi/2$ (or $v_1 = 0$, in case of headon collision)

For inelastic collision:

Due to heat loss, energy equation becomes:

$v_0^2 > v_1^2 + v_2^2$ (2) $\therefore$ From 1,2: $\cos \theta > 0 \Rightarrow 0 \leq \theta < \pi/2$

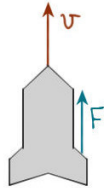
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1.178. A rocket ejects a steady jet whose velocity is equal to $\vec{u}$ relative to the rocket. The gas discharge rate equals μ kg/s. Demonstrate that the rocket motion equation in this case takes the form

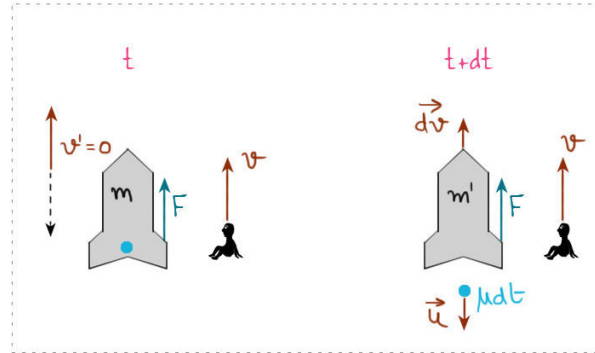
$$m\vec{w} = \vec{F} - \mu\vec{u},$$

where m is the mass of the rocket at a given moment, $\vec{w}$ is its acceleration, and $\vec{F}$ is the external force.

In ground frame
at time t



In reference frame



Impulse = change in momentum:

$$\begin{aligned} \vec{F} dt &= (m' d\vec{v} + \mu dt \vec{u}) \\ \Rightarrow \vec{F} &= m \frac{d\vec{v}}{dt} + \mu \vec{u} \\ \Rightarrow m \vec{w} &= \vec{F} - \mu \vec{u} \end{aligned}$$

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1.179. A rocket moves in the absence of external forces by ejecting a steady jet with velocity $\vec{u}$ constant relative to the rocket. Find the velocity $\vec{v}$ of the rocket at the moment when its mass is equal to m , if at the initial moment it possessed the mass m_0 and its velocity was equal to zero. Make use of the formula given in the foregoing problem.

$$m \vec{a} = \vec{F}_{\text{ext}} - \mu \vec{u}$$

$$\begin{aligned} m \vec{a} &= \vec{F}_{\text{ext}} - \mu \vec{u} \\ \Rightarrow m \frac{d\vec{v}}{dt} &= - \left(-\frac{dm}{dt} \right) \vec{u} \quad \text{-ve as } \mu > 0 \text{ \& } \frac{dm}{dt} < 0 \\ \Rightarrow \int_0^{\vec{v}} d\vec{v} &= \vec{u} \int_{m_0}^m \frac{dm}{m} \\ \Rightarrow \vec{v} &= \vec{u} \ln \frac{m}{m_0} \end{aligned}$$

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1.180. Find the law according to which the mass of the rocket varies with time, when the rocket moves with a constant acceleration w , the external forces are absent, the gas escapes with a constant velocity u relative to the rocket, and its mass at the initial moment equals m_0 .

$$m \frac{d\vec{v}}{dt} = -u \vec{u}$$

$$\Rightarrow m \vec{a} = - \left(- \frac{dm}{dt} \right) \vec{u}$$

$$\vec{a} \int_0^t dt = \vec{u} \int_{m_0}^m \frac{dm}{m}$$

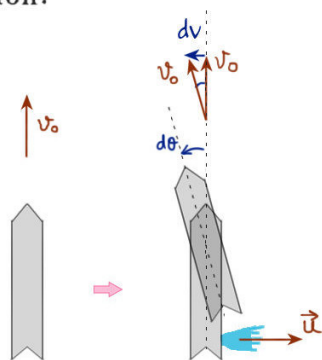
$$\Rightarrow \vec{a} t = \vec{u} \ln \frac{m}{m_0}$$

$$\Rightarrow \omega t = (-u) \ln \frac{m}{m_0} \quad (\text{as } \vec{a} \text{ \& } \vec{u} \text{ are in opposite directions})$$

$$\Rightarrow m = m_0 e^{-\omega t / u} //$$

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1.181. A spaceship of mass m_0 moves in the absence of external forces with a constant velocity $\vec{v}_0$. To change the motion direction, a jet engine is switched on. It starts ejecting a gas jet with velocity u which is constant relative to the spaceship and directed at right angles to the spaceship motion. The engine is shut down when the mass of the spaceship decreases to m . Through what angle α did the motion direction of the spaceship deviate due to the jet engine operation?



As the force by eject gases is perpendicular to motion, v_0 is unchanged.

$$\text{We know, } m \frac{d\vec{v}}{dt} = \vec{u} \frac{dm}{dt}$$

Writing magnitudes and making both sides +ve

$$m dv = u (-dm)$$

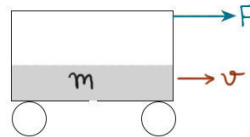
$$v_0 \int_0^\alpha d\theta = -u \int_{m_0}^m \frac{dm}{m}$$

$$\Rightarrow \alpha = \frac{u}{v_0} \ln \frac{m_0}{m} //$$

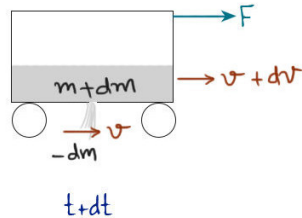
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1.182. A cart loaded with sand moves along a horizontal plane due to a constant force F coinciding in direction with the cart's velocity vector. In the process, sand spills through a hole in the bottom with a constant velocity μ kg/s. Find the acceleration and the velocity of the cart at the moment t , if at the initial moment $t = 0$ the cart with loaded sand had the mass m_0 and its velocity was equal to zero. The friction is to be neglected.

Method 1 wrt ground



At general time t



Impulse = change in momentum

$$F dt = [(m+dm)(v+dv) + (-dm)v] - mv$$

$$\Rightarrow F dt = m dv + dm dv \approx 0$$

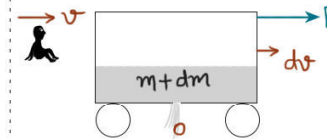
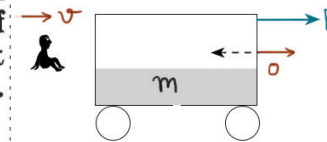
$$F = m \frac{dv}{dt}$$

$$m = m_0 - \mu t$$

$$\Rightarrow F \int_0^t \frac{dt}{m_0 - \mu t} = \int_0^v dv \Rightarrow v = \frac{F}{\mu} \ln \left(\frac{m_0}{m_0 - \mu t} \right)$$

Method 2

wrt frame as shown

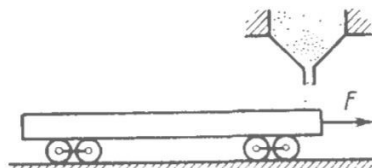


$$J = p_f - p_i$$

$$F dt = (m+dm) dv - 0$$

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1.183. A flatcar of mass m_0 starts moving to the right due to a constant horizontal force F (Fig. 1.46). Sand spills on the flatcar from a stationary hopper. The velocity of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flatcar in the process of loading. The friction is negligibly small.



At time t : m, v

At time $t + dt$: $m + dm, v + dv$

Impulse = change in momentum

$$F dt = (m+dm)(v+dv) - mv$$

$$= m dv + v dm + dm dv \approx 0$$

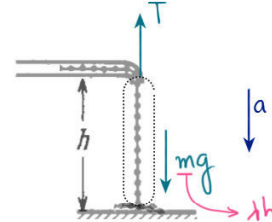
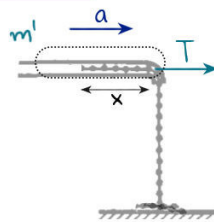
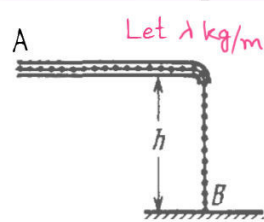
$$\Rightarrow F dt = d(mv)$$

$$\Rightarrow F \int_0^t dt = \int_0^v d(mv) \quad (\text{as } m = m_0 + \mu t)$$

$$\Rightarrow v = \frac{Ft}{m_0 + \mu t}$$

Youtube / Arihant Senior

1.184. A chain AB of length l is located in a smooth horizontal tube so that its fraction of length h hangs freely and touches the surface of the table with its end B (Fig. 1.47). At a certain moment the end A of the chain is set free. With what velocity will this end of the chain slip out of the tube?



From 1,2:

$$\begin{aligned} hg - xa &= ha \\ \Rightarrow hg &= (x+h)a \end{aligned}$$

$$\Rightarrow hg = (x+h) \frac{v dv}{(-dx)}$$

($\because dx$ is -ve, but a is +ve, we make the whole term +ve with a - sign)

$$\begin{aligned} T &= m'a \\ (\lambda x) a & \quad (1) \end{aligned}$$

$$(\lambda h)g - T = (\lambda h)a \quad (2)$$

$$\Rightarrow -gh \int_{l-h}^0 \frac{dx}{x+h} = \int_0^v v dv \Rightarrow v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$

Youtube / Arihant Senior

1.185. The angular momentum of a particle relative to a certain point O varies with time as $\vec{L} = \vec{a} + \vec{b}t^2$, where $\vec{a}$ and $\vec{b}$ are constant vectors, with $\vec{a} \perp \vec{b}$. Find the force moment $\vec{\tau}$ relative to the point O acting on the particle when the angle between the vectors $\vec{L}$ and $\vec{\tau}$ equals 45° .

Given that:

$$\begin{aligned} \vec{L} &= \vec{a} + \vec{b}t^2 \\ \Rightarrow \vec{\tau} &= \frac{d\vec{L}}{dt} = 2\vec{b}t \end{aligned}$$

$$\begin{aligned} \cos 45^\circ &= \frac{\vec{L} \cdot \vec{\tau}}{|\vec{L}| |\vec{\tau}|} = \frac{1}{\sqrt{2}} \\ \Rightarrow \frac{2b^2 t^3}{\sqrt{a^2 + b^2 t^4} (2bt)} &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\Rightarrow 2b^2 t^4 = a^2 + b^2 t^4$$

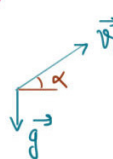
$$\Rightarrow t^4 = \frac{a^2}{b^2}$$

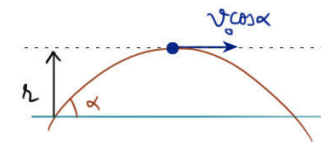
$$\Rightarrow t = \sqrt{\frac{a}{b}}$$

$$\tau|_{t=\sqrt{\frac{a}{b}}} = 2\sqrt{\frac{a}{b}} \vec{b}$$

Youtube / Arihant Senior

1.186. A ball of mass m is thrown at an angle α to the horizontal with the initial velocity v_0 . Find the time dependence of the magnitude of the ball's angular momentum vector relative to the point from which the ball is thrown. Find the angular momentum M at the highest point of the trajectory if $m = 130$ g, $\alpha = 45^\circ$, and $v_0 = 25$ m/s. The air drag is to be neglected.

$$\begin{aligned}
 L &= m (\vec{r} \times \vec{v}) \\
 &= m (\vec{v}_0 t + \frac{1}{2} \vec{g} t^2) \times (\vec{v}_0 + \vec{g} t) \\
 &= m (\vec{v}_0 \times \vec{g}) \frac{t^2}{2} \\
 &= m \frac{v_0 g t^2}{2} \cos \alpha \quad \text{//}
 \end{aligned}$$




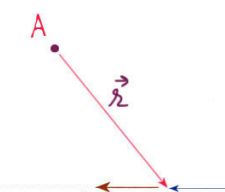
$$\begin{aligned}
 L &= m v r_{\perp} \\
 \text{At highest point: } v_0 \cos \alpha \quad H &= \frac{v_0^2 \sin^2 \alpha}{2g} \\
 &= \frac{m v_0^3 \sin^2 \alpha \cos \alpha}{2g} \quad \text{//}
 \end{aligned}$$

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1.187. A disc A of mass m sliding over a smooth horizontal surface with velocity v experiences a perfectly elastic collision with a smooth stationary wall at a point O (Fig. 1.48). The angle between the motion direction of the disc and the normal of the wall is equal to α . Find:

(a) the points relative to which the angular momentum M of the disc remains constant in this process;

(b) the magnitude of the increment of the vector of the disc's angular momentum relative to the point O' which is located in the plane of the disc's motion at the distance l from the point O .

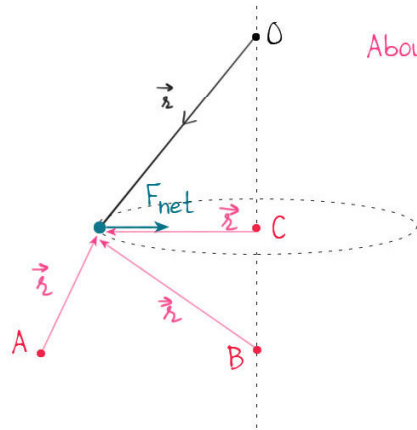
$$\begin{aligned}
 \text{B } \Delta \vec{L} &= \vec{r} \times \vec{J} \\
 &= 2mv l \cos \alpha \hat{k} \quad \text{//}
 \end{aligned}$$


$$\begin{aligned}
 \text{A } &\text{We know, about a point} \\
 &\vec{L} = \vec{r} \times \vec{p} \\
 \Rightarrow \Delta \vec{L} &= \vec{r} \times \Delta \vec{p} \\
 &= \vec{r} \times \vec{J}
 \end{aligned}$$

Non-zero about every point (ex. A)
except ones on XX' (ex. B)

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1.188. A small ball of mass m suspended from the ceiling at a point O by a thread of length l moves along a horizontal circle with a constant angular velocity ω . Relative to which points does the angular momentum $\vec{M}$ of the ball remain constant? Find the magnitude of the increment of the vector of the ball's angular momentum relative to the point O picked up during half a revolution.



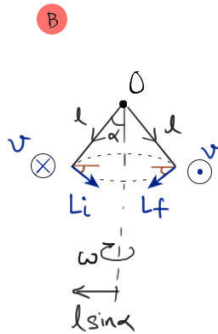
A $\vec{z} = \vec{r} \times \vec{F}$

About point A: $\vec{z}$ (and therefore $\vec{L}$) changes direction and magnitude

B: $\vec{z}$ (and therefore $\vec{L}$) changes only direction.

C: $\vec{z}$ is zero. $\Rightarrow \vec{L}$ is constant //

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$\vec{L} = m \vec{r} \times \vec{v}$

About O, during half revolution, L changes direction by 90°

$|\vec{L}_f - \vec{L}_i| = 2L_f \cos \alpha$

$= 2(mvl) \cos \alpha$
 $\quad \quad \quad \omega l \sin \alpha$

$= 2m\omega l^2 \sin \alpha \cos \alpha$

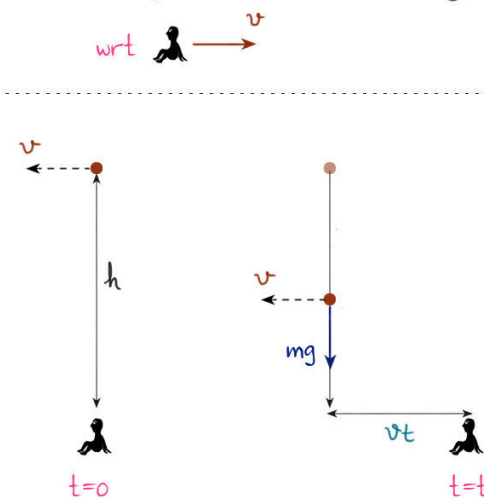
$= 2m\omega l^2 \sqrt{1 - \left(\frac{g}{\omega^2 l}\right)^2} \cdot \frac{g}{\omega^2 l}$

$= \frac{2mlg}{\omega} \sqrt{1 - \left(\frac{g}{\omega^2 l}\right)^2}$ //

$\left\{ \begin{array}{l} mg = T \cos \alpha \quad (1) \\ T \sin \alpha = m\omega^2 l \sin \alpha \quad (2) \end{array} \right.$
 From 1, 2:
 $\cos \alpha = \frac{g}{\omega^2 l}$

Youtube / Arihant Senior

1.189. A ball of mass m falls down without initial velocity from a height h over the Earth's surface. Find the increment of the ball's angular momentum vector picked up during the time of falling (relative to the point O of the reference frame moving translationally in a horizontal direction with a velocity V). The ball starts falling from the point O . The air drag is to be neglected.



$$\begin{aligned}
 dL &= \tau dt \\
 &= (v \cdot t) mg \\
 &= mgv t dt \\
 \Rightarrow L &= \int_0^L dL = mgv \int_0^{\sqrt{\frac{2h}{g}}} t dt \\
 \Rightarrow L &= \frac{mgv}{2} \cdot \frac{2h}{g} = mvh //
 \end{aligned}$$

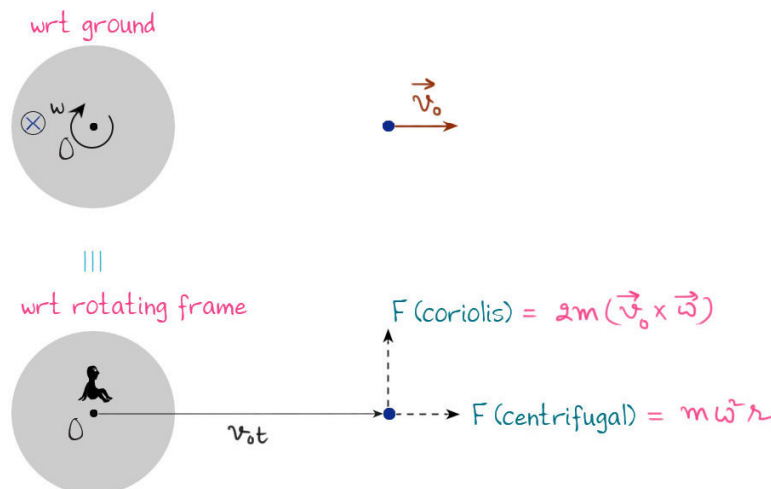
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1.190. A smooth horizontal disc rotates with a constant angular velocity ω about a stationary vertical axis passing through its centre, the point O . At a moment $t = 0$ a disc is set in motion from that point with velocity v_0 . Find the angular momentum $M(t)$ of the disc relative to the point O in the reference frame fixed to the disc. Make sure that this angular momentum is caused by the Coriolis force.

Coriolis Force is a pseudo force, given by:

$$F_{\text{cor}} = 2m(\vec{v} \times \vec{\omega})$$

v of particle, wrt axis of rotation



$$\begin{aligned}
 \therefore dL &= \tau dt \\
 &= F_{\text{cor}} \cdot (v_0 t) \\
 &= 2m v_0 \omega v_0 t dt \\
 \Rightarrow L &= 2m v_0^2 \omega \int_0^t t dt \\
 &= m v_0^2 \omega t^2 //
 \end{aligned}$$

Youtube / Arihant Senior

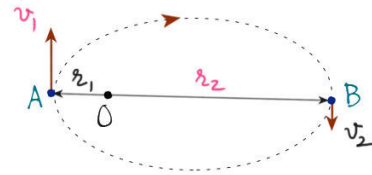
1.191. A particle moves along a closed trajectory in a central field of force where the particle's potential energy $U = kr^2$ (k is a positive constant, r is the distance of the particle from the centre O of the field). Find the mass of the particle if its minimum distance from the point O equals r_1 and its velocity at the point farthest from O equals v_2 .

As force is central in nature, its conservative. ①
Besides, potential is defined only for a conservative force.

$$F = -\frac{dU}{dr} \hat{r}$$

$$\Rightarrow F = -2kr \hat{r}$$

-ve sign implies attractive force ②

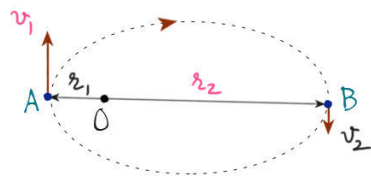


- Force is always towards O .
 > Torque about O is zero.
 > Angular momentum about O is constant. ③

At nearest and farthest points, normal velocity: $dr/dt = 0$
 particle will only have tangential velocity

> $\vec{v} \perp \vec{r}$ ④

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Momentum conservation (A and B):

$$vr = \text{constant}$$

↓ diff..

$$vdr + r dv = 0$$

$$\Rightarrow vdr = -r dv \quad ①$$

Energy conservation (general case):

$$U + KE = kr^2 + \frac{1}{2}mv^2 = \text{const}$$

↓ diff..

$$2kr dr + mvdv = 0$$

$$\Rightarrow 2kr dr = -mvdv \quad ②$$

$$\frac{①}{②} \Rightarrow \frac{v}{2kr} = \frac{-r}{-mv}$$

$$\Rightarrow mv^2 = 2kr^2 \quad ③$$

Energy conservation (A and B):

$$\frac{1}{2}mv_1^2 + kr_1^2 = \frac{1}{2}mv_2^2 + kr_2^2$$

$\downarrow$ $\frac{2kr_1^2}{m}$ ③ $\downarrow$ $\frac{mv_2^2}{2k}$ ③

$$\Rightarrow 2kr_1^2 = mv_2^2$$

$$\Rightarrow m = \frac{2kr_1^2}{v_2^2} //$$

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If axis and $\vec{F}_{\text{net}}$ are always in same plane, then L is constant about that axis. (as torque about axis is zero in direction of axis itself)

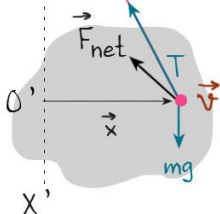
ex. motion of ball hanging by a thread.

$\vec{L}_{XX'} = \text{Component of } \vec{L}_O \text{ about } XX' = 0 \quad (\because \vec{\tau}_O \perp XX')$

$\vec{\tau}_O = \vec{r} \times \vec{F}_{\text{net}}$

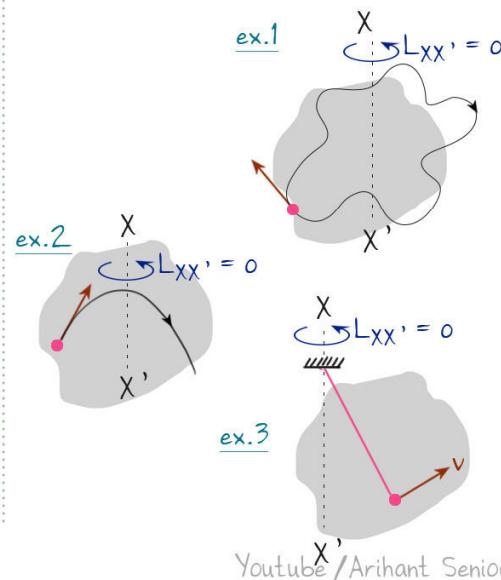
$\Rightarrow \vec{L}_{XX'} = \text{Component of } \vec{L}_O \text{ about } XX' = \text{constant}$

$\vec{L}_O = m(\vec{r} \times \vec{v})$



Note: O' can be taken anywhere on XX' . Torque about it will still be perpendicular to plane.

Special case: if axis and $\vec{v}$ are always in same plane (implying $\vec{F}$ too is always in same plane), then L is always zero about that axis.



Youtube / Arihant Senior

1.192. A small ball is suspended from a point O by a light thread of length l . Then the ball is drawn aside so that the thread deviates through an angle θ from the vertical and set in motion in a horizontal direction at right angles to the vertical plane in which the thread is located. What is the initial velocity that has to be imparted to the ball so that it could deviate through the maximum angle $\pi/2$ in the process of motion?

Angular momentum conservation:

$$\vec{L}_{XX'} = 0$$

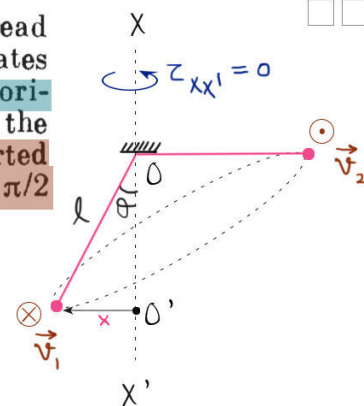
$\Rightarrow \vec{L}_{XX'} = \text{Component of } \vec{L}_O \text{ about } XX' = \text{constant}$

$\vec{L}_O = m(\vec{r} \times \vec{v})$

$\therefore m r v_1 = m l v_2$

$l \sin \theta$

$\Rightarrow v_1 \sin \theta = v_2 \quad (1)$



Energy conservation:

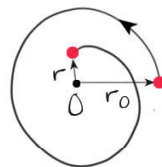
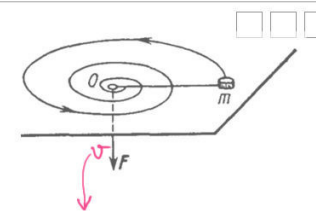
$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_2^2 + m g l \cos \theta \quad (2)$$

From 1,2:

$$v_1 = \sqrt{\frac{2gl}{\cos \theta}}$$

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1.193. A small body of mass m tied to a non-stretchable thread moves over a smooth horizontal plane. The other end of the thread is being drawn into a hole O (Fig. 1.49) with a constant velocity. Find the thread tension as a function of the distance r between the body and the hole if at $r = r_0$ the angular velocity of the thread is equal to ω_0 .



Tension is always directed towards O

$$\therefore \tau_O = 0$$

$$\Rightarrow L_O = \text{constant}$$

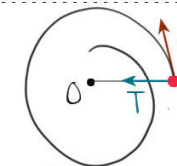
$$\Rightarrow m\omega_0 r_0^2 = m\omega r^2 \quad (1)$$

Acceleration towards O : $\frac{d^2 r}{dt^2} + \omega^2 r$ (as its given, $dr/dt = \text{constant}$)

$$\therefore T = m\omega^2 r \quad (2)$$

$$\text{From 1,2: } T = m \left(\frac{\omega_0 r_0^2}{r^2} \right)^2 r$$

$$= \frac{m\omega_0^2 r_0^4}{r^3} //$$



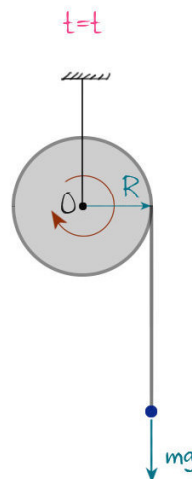
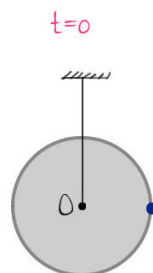
Note: In this problem, thread/Tension is not perpendicular to velocity (as trajectory is spiral, not circular). So a_0 is not a_n and v is not ωr .

$$\text{and } L_O = m\omega r^2 \neq \frac{mv^2}{\omega}$$

Youtube/Arihant Senior

1.194. A light non-stretchable thread is wound on a massive fixed pulley of radius R . A small body of mass m is tied to the free end of the thread. At a moment $t = 0$ the system is released and starts moving. Find its angular momentum relative to the pulley axle as a function of time t .

Not massless!



On the pulley-body system, only torque is due to mg .

$\therefore$ Angular impulse = change in angular momentum about O

$$\tau t = (L_f - L_i)$$

$$(mgR)t = L_f$$

$$\Rightarrow L_f = mgRt //$$

1.195. A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane at an angle α to the horizontal. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the obtained result change in the case of a perfectly smooth inclined plane?



With friction

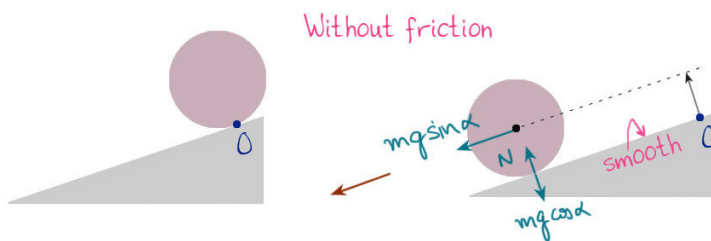
Angular impulse = Change in angular momentum

$$\text{i.e. } \int dt = dL$$

Of the four forces on rolling sphere, only $mg \sin \alpha$ will exert torque about O.

$$\therefore (mg \sin \alpha R) \int_0^t dt = \int_0^L dL$$

$$\Rightarrow L = mg \sin \alpha R t$$



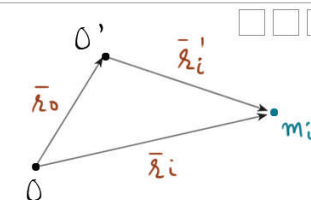
Without friction

On smooth plane, torque is still due to only $mg \sin \alpha$

$\therefore L$ is same

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1.196. A certain system of particles possesses a total momentum $\vec{p}$ and an angular momentum $\vec{L}$ relative to a point O. Find its angular momentum $\vec{L}'$ relative to a point O' whose position with respect to the point O is determined by the radius vector $\vec{r}_0$. Find out when the angular momentum of the system of particles does not depend on the choice of the point O.



Lets consider m_i as one of the many system of particles.

$$\vec{L} = \sum m_i (\vec{r}_i \times \vec{v}_i) \quad (1)$$

$$\vec{L}' = \sum m_i (\vec{r}'_i \times \vec{v}'_i) \quad \left\{ \begin{array}{l} \vec{v}'_i = \frac{d\vec{r}'_i}{dt} \\ = \frac{d}{dt} (\vec{r}_i - \vec{r}_0) \\ = \vec{v}_i \end{array} \right.$$

$$= \sum m_i (\vec{r}_i - \vec{r}_0) \times \vec{v}_i$$

$$= \underbrace{\sum m_i (\vec{r}_i \times \vec{v}_i)}_{\vec{L}} - \vec{r}_0 \times \underbrace{\sum m_i \vec{v}_i}_{\vec{p}}$$

$$= \vec{L} - \vec{r}_0 \times \vec{p} \quad (2)$$

From 2, If O' is on center of mass:

$$\vec{L}'_{cm} = \vec{L} - \vec{r}_{0cm} \times \vec{p}$$

(constant)

$\therefore$ If secondary reference frame (i.e O') is on CoM, then L_{cm} being a constant is independent of L (and therefore choice of O)

Youtube / Arihant Senior

1.197. Demonstrate that the angular momentum $\vec{L}$ of the system of particles relative to a point O of the reference frame K can be represented as

$$\vec{L} = \tilde{\vec{L}} + [\vec{r}_c \vec{p}],$$

where $\tilde{\vec{L}}$ is its proper angular momentum (in the reference frame moving translationally and fixed to the centre of inertia), $\vec{r}_c$ is the radius vector of the centre of inertia relative to the point O , $\vec{p}$ is the total momentum of the system of particles in the reference frame K .

$$\vec{L} = \tilde{\vec{L}} + [\vec{r}_c \vec{p}],$$

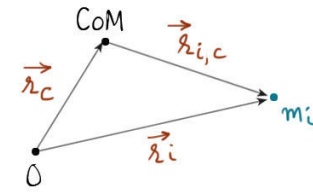
Where,

$$\vec{L} = \sum m_i (\vec{r}_i \times \vec{v}_i) \quad (\text{L of SoP wrt } O)$$

$$\tilde{\vec{L}} = \sum m_i (\vec{r}_{i,c} \times \vec{v}_{i,c}) \quad (\text{L of SoP wrt CoM})$$

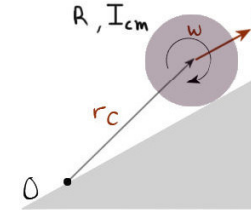
$$\vec{r}_c = \vec{r}_c \quad (\text{position of CoM wrt } O)$$

$$\vec{p} = \sum m_i \vec{v}_i \quad (\text{p of SoP wrt } O)$$



Application:

$$L_O = ?$$



$$L_{\text{sphere}, O} = L_{\text{sphere}, \text{CoM}} + \vec{r}_c \times (m\vec{v})$$

$$\Rightarrow L_{\text{sphere}, O} = I_{cm} \omega + m v R$$

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1.197. Demonstrate $\vec{L} = \tilde{\vec{L}} + [\vec{r}_c \vec{p}]$,

Sol: Lets consider m_i as one of the many system of particles.

$$\vec{L} = \sum m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum m_i (\vec{r}_c + \vec{r}_{i,c}) \times \vec{v}_i$$

$$= \sum m_i (\vec{r}_c \times \vec{v}_i) + \sum m_i (\vec{r}_{i,c} \times \vec{v}_i)$$

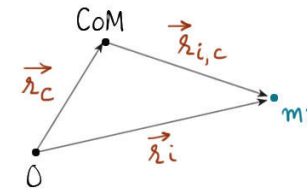
$$= \vec{r}_c \times \sum m_i \vec{v}_i + \sum m_i [\vec{r}_{i,c} \times (\vec{v}_c + \vec{v}_{i,c})]$$

$$\quad \quad \quad \downarrow$$

$$\vec{p}$$

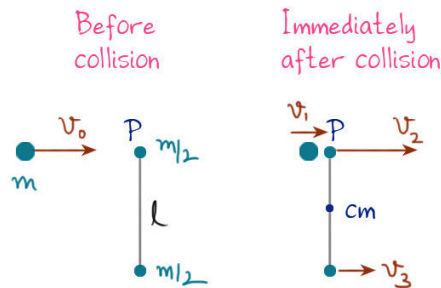
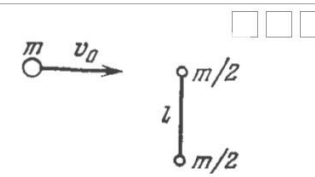
$$= \vec{r}_c \times \vec{p} + \underbrace{(\sum m_i \vec{r}_{i,c}) \times \vec{v}_c}_0 + \underbrace{\sum m_i (\vec{r}_{i,c} \times \vec{v}_{i,c})}_{\tilde{\vec{L}}}$$

$$= \tilde{\vec{L}} + \vec{r}_c \times \vec{p} //$$



Youtube / Arihant Senior

1.198. A ball of mass m moving with velocity v_0 experiences a head-on elastic collision with one of the spheres of a stationary rigid dumbbell as shown in Fig. 1.50. The mass of each sphere equals $m/2$, and the distance between them is l . Disregarding the size of the spheres, find the proper angular momentum $\tilde{L}$ of the dumbbell after the collision, i.e. the angular momentum in the reference frame moving translationally and fixed to the dumbbell's centre of inertia.



Conservation of LM: $mv_0 = mv_1 + \frac{m}{2}v_2 + \frac{m}{2}v_3$ (1)

$V_{os} = V_{oa}$: $v_0 = v_2 - v_1$ (2)

Conservation of AM about point of collision P: $0 = \frac{m}{2}v_3l \Rightarrow v_3 = 0$ (3)

From 1,2: $v_2 = \frac{4v_0}{3}$

$$L_{cm} = \frac{m}{2}v_2 \frac{l}{2} - \frac{m}{2}v_3 \frac{l}{2}$$

$$= \frac{mv_0 l}{3}$$

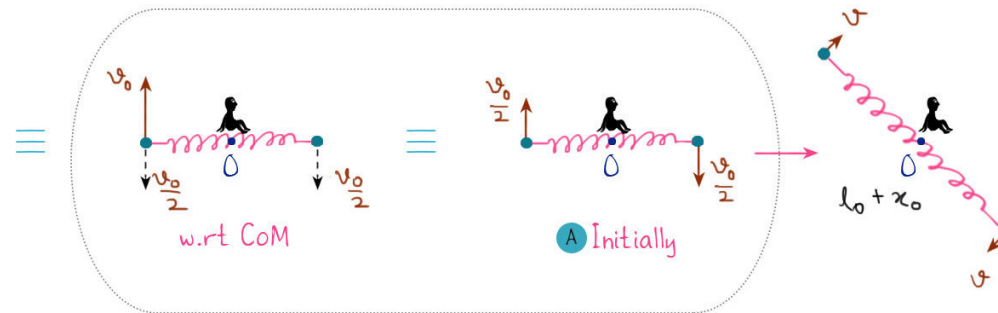
Note: v_3 is zero qualitatively too, as horizontal velocity gained by top ball doesn't immediately increase tension in the rod

Youtube/Arihant Senior

1.199. Two small identical discs, each of mass m , lie on a smooth horizontal plane. The discs are interconnected by a light non-deformed spring of length l_0 and stiffness k . At a certain moment one of the discs is set in motion in a horizontal direction perpendicular to the spring with velocity v_0 . Find the maximum elongation of the spring in the process of motion, if it is known to be considerably less than unity.

B At maximum elongation

$$v_{cm} = \frac{mv_0}{m+m} = \frac{v_0}{2}$$



Conserving Angular momentum at A and B:

$$2\left(\frac{mv_0}{2} \frac{l_0}{2}\right) = 2\left(mv \frac{l_0 + x_0}{2}\right) \quad (1)$$

Conserving Energy

$$2\left(\frac{1}{2}m\left(\frac{v_0}{2}\right)^2\right) = 2\left(\frac{1}{2}mv^2\right) + \frac{1}{2}kx_0^2 \quad (2)$$

From 1,2:

$$x_0 \approx \frac{mv_0^2}{k l_0}$$

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$$\cancel{2} \left(\cancel{m} \frac{v_0}{\cancel{2}} \frac{l_0}{\cancel{2}} \right) = \cancel{2} \left(\cancel{m} v \frac{l_0 + x_0}{\cancel{2}} \right) \quad (1) \longrightarrow v_0 l_0 = 2v(l_0 + x_0)$$

$$\cancel{2} \left(\frac{1}{\cancel{2}} m \left(\frac{v_0}{\cancel{2}} \right)^2 \right) = \cancel{2} \left(\frac{1}{\cancel{2}} m v^2 \right) + \frac{1}{2} k x_0^2 \quad (2) \longrightarrow 4mv^2 + 2kx_0^2 = mv_0^2$$

Solving 1 and 2:

$$\Rightarrow \cancel{4} m \frac{v_0^2 l_0^2}{\cancel{4} (l_0 + x_0)^2} + 2kx_0^2 = mv_0^2$$

$$\Rightarrow \frac{mv_0^2 l_0^2}{l_0^2} \left(1 + \frac{x_0}{l_0} \right)^{-2} + 2kx_0^2 = mv_0^2$$

$$\Rightarrow mv_0^2 \left(1 - \frac{2x_0}{l_0} \right) + 2kx_0^2 = \cancel{m} v_0^2$$

$$\Rightarrow \cancel{2} k x_0^2 = \cancel{m} v_0^2 \frac{2x_0}{l_0}$$

$$\Rightarrow x_0 = \frac{mv_0^2}{kl_0} //$$

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1.200. A planet of mass M moves along a circle around the Sun with velocity $v = 34.9$ km/s (relative to the heliocentric reference frame). Find the period of revolution of this planet around the Sun.

Reference frame with sun as center

$$F = ma$$

$$\frac{GM_s M}{R^2} = M \frac{v^2}{R}$$

$$v = \sqrt{\frac{GM_s}{R}}$$

v-R

$$T = \frac{2\pi R}{v} \quad R = \frac{GM_s}{v^2} \longrightarrow = \frac{2\pi \frac{GM_s}{v^2}}{v}$$

$$v = \sqrt{\frac{GM_s}{R}}$$

$$= \frac{2\pi R}{\sqrt{\frac{GM_s}{R}}}$$

$$\Rightarrow T = \frac{2\pi GM_s}{v^3} = 225 \text{ days} //$$

T-v

$$\Rightarrow T = \frac{2\pi R^{3/2}}{\sqrt{GM_s}}$$

T-R

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1.201. The Jupiter's period of revolution around the Sun is 12 times that of the Earth. Assuming the planetary orbits to be circular, find:

(a) how many times the distance between the Jupiter and the Sun exceeds that between the Earth and the Sun;

(b) the velocity and the acceleration of Jupiter in the heliocentric reference frame.

A $T^2 \propto R^3$

$$\Rightarrow \left(\frac{T_J}{T_E}\right)^{2/3} = \frac{R_J}{R_E}$$

$$\Rightarrow \frac{R_J}{R_E} = 12^{2/3} //$$

B $T_J = \frac{2\pi G M_S}{v_J^3}$

$$\Rightarrow v_J = \left(\frac{2\pi G M_S}{T_J}\right)^{1/3}$$

$$= 13 \text{ km/s} //$$

$$a_J = \frac{v_J^2}{R_J}$$

$$= \frac{v_J^2}{\left(\frac{\sqrt{G M_S T_J}}{2\pi}\right)^{2/3}}$$

$$= 2.2 \times 10^{-4} \text{ m/s}^2 //$$

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1.202. A planet of mass M moves around the Sun along an ellipse so that its minimum distance from the Sun is equal to r and the maximum distance to R . Making use of Kepler's laws, find its period of revolution around the Sun.

Keplers three laws:

- 1 The orbit of a planet is an ellipse with the Sun at one of the two foci.
- 2 A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- 3 The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

Sol: From 3, for a planets elliptical orbit: $T_P^2 \propto \left(\frac{R+r}{2}\right)^3$

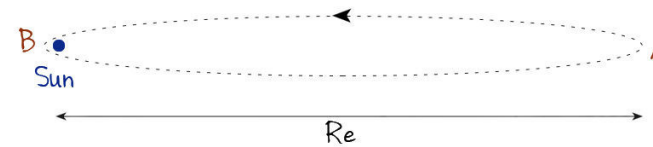
From 3, for circular orbit of same T_P : $T_P^2 \propto \left(\frac{R'+R'}{2}\right)^3 = R'^3 \Rightarrow R' = \frac{R+r}{2}$

We know T_P of a circular orbit is given by: $T_P = \frac{2\pi R'^{3/2}}{\sqrt{G M_S}} \Rightarrow T_P = \frac{2\pi \left(\frac{R+r}{2}\right)^{3/2}}{\sqrt{G M_S}} //$

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1.203. A small body starts falling onto the Sun from a distance equal to the radius of the Earth's orbit. The initial velocity of the body is equal to zero in the heliocentric reference frame. Making use of Kepler's laws, find how long the body will be falling.

Directly falling body can be considered to be orbiting the sun in a elongated ellipse, if we assume sun to be a point object, with focus near the end.



∴ If time period of orbit is T , Time of fall = $T/2$

$R + r \approx R_e$

From Kepler's third law $T^2 \propto \left(\frac{R+r}{2}\right)^3 \Rightarrow T^2 \propto \left(\frac{R_e}{2}\right)^3$
for the body:

Also Kepler's third law $T_e^2 \propto \left(\frac{R_e + R_e}{2}\right)^3 \Rightarrow T_e^2 \propto R_e^3$
for earth:

$$\frac{T}{T_e} = \left(\frac{1}{2}\right)^{3/2} \Rightarrow T = \frac{T_e}{2^{3/2}}$$

$$\Rightarrow T_{\text{fall}} = \frac{T}{2} = \frac{T_e}{2 \cdot 2^{3/2}} = 64.5 \text{ days} //$$

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1.204. Suppose we have made a model of the Solar system scaled down in the ratio η but of materials of the same mean density as the actual materials of the planets and the Sun. How will the orbital periods of revolution of planetary models change in this case?

We know, $T = \frac{2\pi R^{3/2}}{\sqrt{GM_s}}$

$\Rightarrow T \propto \frac{R^{3/2}}{R_s^{3/2}}$

$\frac{4}{3}\pi R_s^3 \rho_s$

$R \rightarrow$ Orbit radius

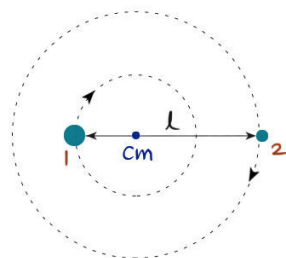
$R_s \rightarrow$ Sun radius

$$\Rightarrow T \propto \left(\frac{R}{R_s}\right)^{3/2}$$

$$\therefore \frac{T \text{ for solar system}}{T \text{ for scaled model}} = \frac{T_1}{T_2} = \frac{\left(\frac{R_1}{R_s}\right)^{3/2}}{\left(\frac{R_1/\eta}{R_s/\eta}\right)^{3/2}} = 1 \Rightarrow T_1 = T_2 //$$

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1.205. A double star is a system of two stars moving around the centre of inertia of the system due to gravitation. Find the distance between the components of the double star, if its total mass equals M and the period of revolution T .



System will move around center of mass.

$$\therefore r_1 = \frac{m_2 l}{m_1 + m_2}$$

On 1, $F = ma$

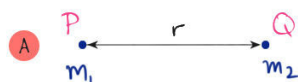
$$\frac{G m_1 m_2}{l^2} = m_1 \omega^2 r_1$$

$$= m_1 \left(\frac{2\pi}{T} \right)^2 \cdot \frac{m_2 l}{m_1 + m_2}$$

$$\Rightarrow l = \left(\frac{G M T^2}{4\pi^2} \right)^{1/3}$$

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- 1.206. Find the potential energy of the gravitational interaction
- (a) of two mass points of masses m_1 and m_2 located at a distance r from each other;
- (b) of a mass point of mass m and a thin uniform rod of mass M and length l , if they are located along a straight line at a distance a from each other; also find the force of their interaction.



$$U_{12} = m_2 V_Q$$

$$dV = -\vec{E} \cdot d\vec{x}$$

$$\Rightarrow \int_{V_Q}^0 dV = - \int_r^\infty \vec{E} \cdot d\vec{x} = \frac{G m_1 m_2}{x^2} dx$$

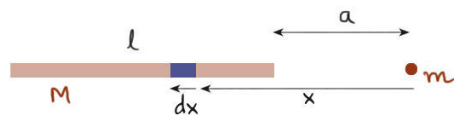
$$\Rightarrow V_Q = -G m_1 \int_r^\infty \frac{dx}{x^2}$$

$$= -G m_1 \left(\frac{1}{r} - \frac{1}{\infty} \right) = -\frac{G m_1}{r}$$

$$\therefore U_{12} = m_2 V_Q = -\frac{G m_1 m_2}{r}$$

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B



$$\begin{aligned}
 dU &= -\frac{G dM m}{x} \\
 &= -\frac{G \left(\frac{dx}{l} M\right) m}{x} \\
 U &= \int_0^U dU = -\frac{G M m}{l} \int_a^{a+l} \frac{dx}{x} \\
 \Rightarrow U &= -\frac{G M m}{l} \ln\left(\frac{a+l}{a}\right) //
 \end{aligned}$$

$$dF = G \left(\frac{dx}{l} M\right) \frac{m}{x^2}$$

$$\begin{aligned}
 F &= \int dF = \frac{G M m}{l} \int_a^{a+l} \frac{dx}{x^2} \\
 &= \frac{G M m}{l} \left(\frac{1}{a} - \frac{1}{a+l}\right) \\
 &= \frac{G M m}{a(a+l)} //
 \end{aligned}$$

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1.207. A planet of mass m moves along an ellipse around the Sun so that its maximum and minimum distances from the Sun are equal to r_1 and r_2 respectively. Find the angular momentum M of this planet relative to the centre of the Sun.

Conservation of angular momentum:

$$m v_1 r_1 = m v_2 r_2 \quad (1)$$

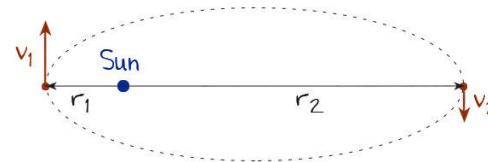
Conservation of energy:

$$\frac{1}{2} m v_1^2 - \frac{G m M_s}{r_1} = \frac{1}{2} m v_2^2 - \frac{G m M_s}{r_2} \quad (2)$$

From 1,2:

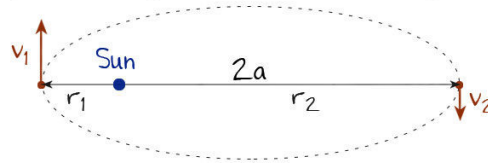
$$v_1 = \sqrt{\frac{2 G M_s r_2}{r_1 (r_1 + r_2)}}$$

$$\Rightarrow L = m v_1 r_1 = m \sqrt{\frac{2 G M_s r_1 r_2}{(r_1 + r_2)}} //$$



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1.208. Using the conservation laws, demonstrate that the total mechanical energy of a planet of mass m moving around the Sun along an ellipse depends only on its semi-major axis a . Find this energy as a function of a .

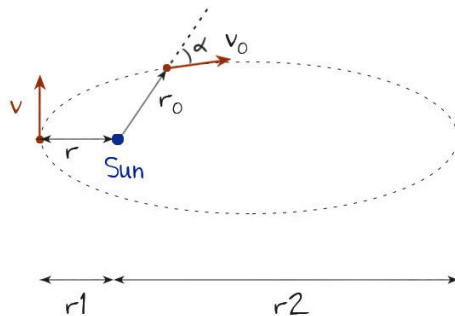


From 1.207: At any one of the end of the ellipse at distance r_1 from sun: $v_1 = \sqrt{\frac{2G m_s r_2}{r_1(r_1+r_2)}}$

$$\begin{aligned} E &= \frac{1}{2} m v_1^2 - \frac{G m m_s}{r_1} \\ &= \frac{1}{2} m \cdot \frac{2G m_s r_2}{r_1(r_1+r_2)} - \frac{G m m_s}{r_1} \\ &= -\frac{G m m_s}{r_1+r_2} = -\frac{G m m_s}{2a} \quad (\text{only depends on } a) \end{aligned}$$

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1.209. A planet A moves along an elliptical orbit around the Sun. At the moment when it was at the distance r_0 from the Sun its velocity was equal to v_0 and the angle between the radius vector r_0 and the velocity vector v_0 was equal to α . Find the maximum and minimum distances that will separate this planet from the Sun during its orbital motion.



With any one of the extreme ends:

Conservation of angular momentum

$$v r = v_0 r_0 \sin \alpha \quad (1)$$

Conservation of energy:

$$\frac{1}{2} m v^2 - \frac{G m m_s}{r} = \frac{1}{2} m v_0^2 - \frac{G m m_s}{r_0} \quad (2)$$

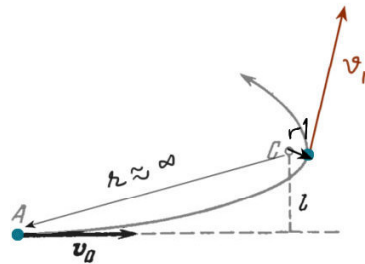
From 1,2:

$$r = r_0 \left(\frac{1 \pm \sqrt{1 - (2 - \eta) \eta \sin^2 \alpha}}{(2 - \eta)} \right), \quad \eta \rightarrow \frac{v_0^2 r_0}{G m_s}$$

Two roots are r_1 and r_2 , nearest and farthest points.

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1.210. A cosmic body A moves to the Sun with velocity v_0 (when far from the Sun) and aiming parameter l the arm of the vector v_0 relative to the centre of the Sun (Fig. 1.51). Find the minimum distance by which this body will get to the Sun.



Conservation of angular momentum

$$v_0 l = v_1 r_1 \quad (1)$$

Conservation of energy:

$$\frac{1}{2} m v_0^2 - \frac{G m m_s}{r \approx \infty} = \frac{1}{2} m v_1^2 - \frac{G m m_s}{r_1} \quad (2)$$

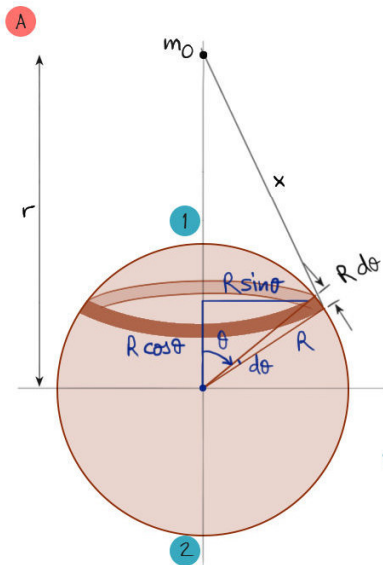
From 1,2:

$$r_1 = \frac{-G m_s \pm \sqrt{G^2 m_s^2 + v_0^4 l^2}}{v_0^2} \quad \text{(Reject -ve solution)}$$

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1.211. A particle of mass m_0 is located outside a uniform sphere of mass M at a distance r from its centre. Find:

- the potential energy of gravitational interaction of the particle and the sphere;
- the gravitational force which the sphere exerts on the particle.



Lets first find U due to a shell of mass M' .
 dU due to the ring element on a shell:

$$dU = -\frac{G m_0 dm}{x} \quad \text{where } dm = \frac{dA}{A} M'$$

$$= \frac{2\pi R \sin \theta R d\theta}{2\pi R^2} \cdot M'$$

$$\Rightarrow U = -\frac{G m_0 M'}{2} \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{R^2 + r^2 - 2rR \cos \theta}}$$

$$= -\frac{G m_0 M'}{2} \Rightarrow U = -\frac{G m_0 M}{2}$$

(Due to shell)

(Due to uniform sphere)

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Solving the integration from previous slide:

$$U_{\text{shell}} = \frac{-G m_0 M'}{2} \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{R^2 + r^2 - 2rR \cos \theta}} \quad \leftarrow \begin{cases} R^2 + r^2 - 2rR \cos \theta = x^2 \\ \Rightarrow 2rR \sin \theta d\theta = 2x dx \end{cases}$$

$$\begin{aligned} \Rightarrow U &= \frac{-G m_0 M'}{2} \int_{R-r}^{R+r} \frac{x dx}{xR} \\ &= \frac{-G m_0 M'}{2} \frac{1}{R} x \Big|_{R-r}^{R+r} \\ &= \frac{-G m_0 M'}{2R} \cancel{2R} = \frac{-G m_0 M'}{R} // \end{aligned}$$

$$\begin{aligned} \text{B } F &= -\frac{dU}{dr} \hat{r} \\ &= -\frac{d}{dr} \left(\frac{-G m_0 M'}{R} \right) \hat{r} \\ &= \frac{G m_0 M'}{R^2} \hat{r} // \end{aligned}$$

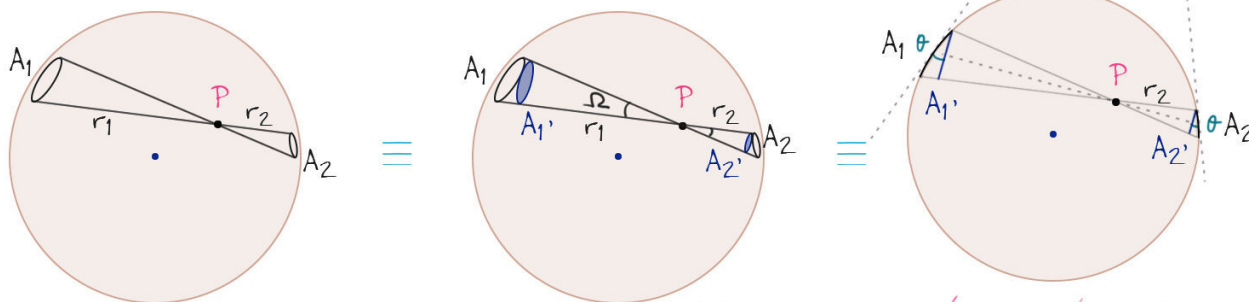
attractive

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1.212. Demonstrate that the **gravitational force** acting on a particle **A** inside a uniform spherical layer of matter is equal to zero.

Method 1

Lets find field at a random point P inside the spherical shell.
Constructing a small angled cone, that cuts the surface of shell.



$$\text{As the solid angle subtended by both the cones is equal: } \frac{A_1'}{r_1^2} = \frac{A_2'}{r_2^2} \Rightarrow \frac{A_1 \cos \theta}{r_1^2} = \frac{A_2 \cos \theta}{r_2^2} \Rightarrow \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$

Due to both elemental areas:

$$dE_P = \frac{G dm_1}{r_1^2} - \frac{G dm_2}{r_2^2} = G(\sigma \frac{A_1}{r_1^2}) - G(\sigma \frac{A_2}{r_2^2}) = 0 \Rightarrow E_P = \int dE_P = 0 //$$

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Method 2 From 1.211:

For a point P outside shell:

$$V_P = -\frac{G m_0 m'}{2} \frac{1}{R} \left[\frac{z+R}{R} - \frac{z-R}{R} \right] \quad \text{--->}$$

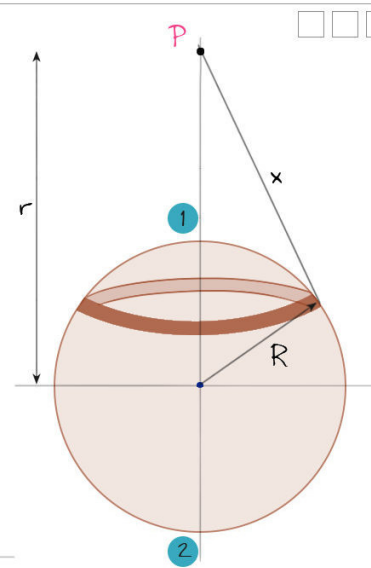
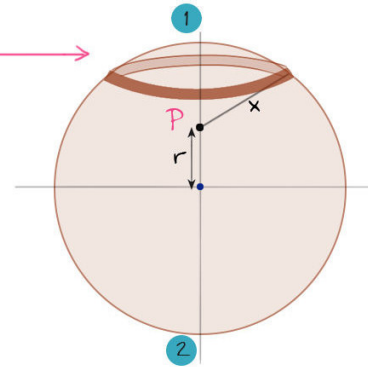
(Potential)

For a point P inside shell:

$$V_P = \left(\quad \right) \left[\frac{R+z}{R} - \frac{R-z}{R} \right]$$

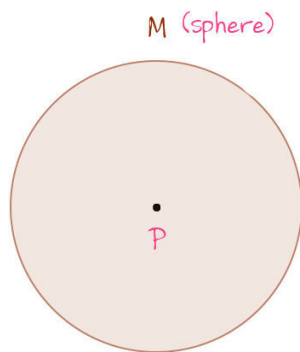
$$= -\frac{G m'}{2/R} \left(\frac{2z}{R} \right) = -\frac{G m'}{R} = \text{constant}$$

$$\Rightarrow E_P = -\frac{dV_P}{dz} \hat{z} = 0 //$$

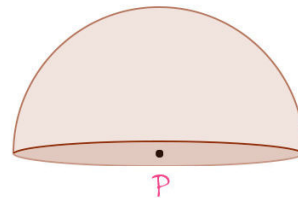


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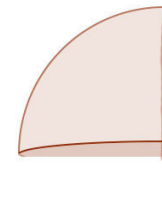
1.213. A particle of mass m was transferred from the centre of the base of a uniform hemisphere of mass M and radius R into infinity. What work was performed in the process by the gravitational force exerted on the particle by the hemisphere?



M (sphere)



M (hemisphere)



M (quartersphere)



M (cone)

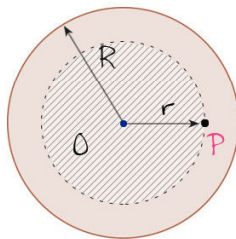
$$V_P \text{ in all four cases} = -\frac{3}{2} \frac{G M}{R} = \text{Wext to bring a unit mass to P}$$

∴ Wext to remove a mass m from P to infinity

$$W_{\text{ext}} = -mV = \frac{3}{2} \frac{G m M}{R} //$$

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1.214. There is a uniform sphere of mass M and radius R . Find the strength G and the potential ϕ of the gravitational field of this sphere as a function of the distance r from its centre (with $r < R$ and $r > R$). Draw the approximate plots of the functions $G(r)$ and $\phi(r)$.

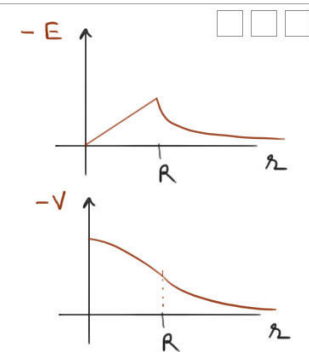


for: $r > R$ $E_{out} = -\frac{GM}{r^2}$, $V_{out} = -\frac{GM}{r}$

$r < R$ $E_{in} = -\frac{GM_{in}}{r^2} = -\frac{G}{r^2} \left(\frac{4}{3}\pi r^3 \rho \right) = -\frac{GM}{R^3} r$

$$V_{in} = \int_{-\frac{GM}{R}}^{V_{in}} dV_{in} = -\int_R^r E \cdot dr \Rightarrow V_{in} = -\frac{GM}{R} - \left(-\frac{GM}{R^3}\right) \int_R^r r dr = -\frac{GM}{R} \left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$$

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1.215. Inside a uniform sphere of density ρ there is a spherical cavity whose centre is at a distance l from the centre of the sphere. Find the strength G of the gravitational field inside the cavity.

Field due to a sphere of given density ρ at its surface:

$$\begin{aligned} \vec{E}_s &= -\frac{GM}{r^2} \hat{r} \\ &= -\frac{G \left(\rho \frac{4}{3}\pi r^3 \right)}{r^2} \hat{r} \\ &= -\frac{4\pi G \rho}{3} r \hat{r} \\ &= -\frac{4\pi G \rho}{3} \vec{r} \end{aligned}$$

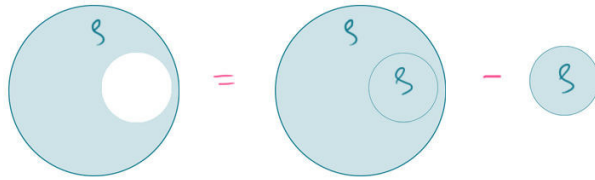
For sphere of total mass M and radius R . $M = \rho \frac{4}{3}\pi R^3$

$$\vec{E}_s = -\frac{GM}{R^3} \vec{r}$$

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Gravitational field inside a spherical cavity within a sphere.

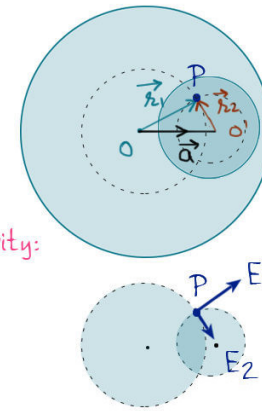
By principle of superposition:



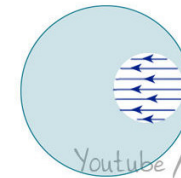
∴ field at any point is now :
Field due to full sphere – field due to small sphere

At a point P inside cavity:

$$\vec{E} = \vec{E}_1 - \vec{E}_2$$

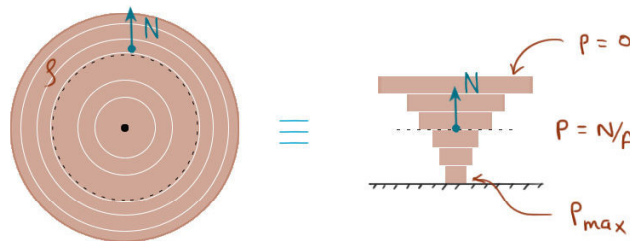


$$= -\frac{4\pi G \rho}{3} (\vec{r}_1 - \vec{r}_2) = -\frac{4\pi G \rho}{3} \vec{a} \quad \text{(A constant)}$$



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1.216. A uniform sphere has a mass M and radius R . Find the pressure p inside the sphere, caused by gravitational compression, as a function of the distance r from its centre. Evaluate p at the centre of the Earth, assuming it to be a uniform sphere.



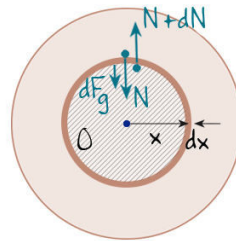
$$p = F/A$$

F is NOT gravitational force, but it's the normal reaction N , that's preventing the outside mass to collapse towards the center.

$$\therefore p = N/A$$

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Calculating N: Writing forces on the spherical elemental shell.



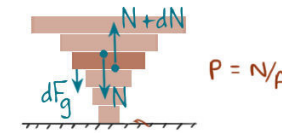
$$N + dN = dF_g + N \quad (\text{As it's at rest})$$

$$dN = dm g_x$$

$$\int 4\pi x^2 dx = \frac{GMx}{R^3}$$

$$\left(\frac{4}{3}\pi R^3\right) \rho = M$$

(derived in 1.215)



$$\Rightarrow dp = \frac{dN}{A} = \frac{dN}{4\pi x^2} = \frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{4\pi x^2 dx}{(4\pi x^2)} \cdot \frac{GMx}{R^3}$$

$$\Rightarrow p = \left| \int_p^0 dp \right| = \frac{3GM^2}{4\pi R^6} \int_0^R x dx = \frac{3GM^2}{8\pi R^6} (R^2 - 0) //$$

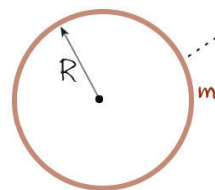
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1.217. Find the proper potential energy of gravitational interaction of matter forming

- (a) a thin uniform spherical layer of mass m and radius R ;
 (b) a uniform sphere of mass m and radius R (make use of the answer to Problem 1.214).

Self energy of shell and sphere

A



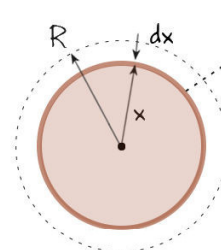
To bring dm mass from infinity, change in energy of the system dU is:

$$dU = dm V$$

$$= dm \left(-\frac{Gm}{R} \right)$$

$$\Rightarrow U_{\text{shell}} = -\frac{G}{R} \int_0^{m_0} m dm = -\frac{G m_0^2}{2R} //$$

B



We bring dm mass from infinity and deposit it on the outermost surface (sphere's radius thus increases by dx). Change in energy of the system dU is:

$$dU = dm V$$

$$= dm \left(-\frac{Gm_x}{x} \right)$$

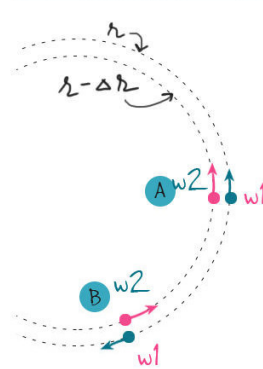
$$m_x = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3} m_0$$

$$= \frac{4\pi x^2 dx}{\frac{4}{3}\pi R^3} m_0$$

$$\Rightarrow U = -\frac{3Gm_0^2}{R^6} \int_0^R x^4 dx \Rightarrow U_{\text{sphere}} = -\frac{3}{5} \frac{Gm_0^2}{R} //$$

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1.218. Two Earth's satellites move in a common plane along circular orbits. The orbital radius of one satellite $r = 7000$ km while that of the other satellite is $\Delta r = 70$ km less. What time interval separates the periodic approaches of the satellites to each other over the minimum distance?



$$\frac{GM_e m}{r^2} = m \omega^2 r$$

$$\Rightarrow \omega = \sqrt{\frac{GM_e}{r^3}}$$

$$\omega_1 = \sqrt{\frac{GM_e}{r^3}}$$

$$\omega_2 = \sqrt{\frac{GM_e}{(r - \Delta r)^3}}$$

$$= \sqrt{\frac{GM_e}{r^3} \left(1 - \frac{\Delta r}{r}\right)^{-3/2}}$$

$$\approx \sqrt{\frac{GM_e}{r^3}} \left(1 + \frac{3\Delta r}{2r}\right)$$

$$\text{Time of separation} = \frac{2\pi R}{v_{\text{rel}}} = \frac{2\pi}{\omega_{\text{rel}}}$$

A They are rotating in similar sense

$$\omega_{\text{rel}} = \omega_2 - \omega_1 \quad \therefore T_1 \approx \frac{2\pi}{\sqrt{\frac{GM_e}{r^3} \left(\frac{3\Delta r}{2r}\right)}}$$

B They are rotating in opposite sense

$$\omega_{\text{rel}} = \omega_2 + \omega_1 \quad \therefore T_2 \approx \frac{2\pi}{\sqrt{\frac{GM_e}{r^3} \left(2 + \frac{3\Delta r}{2r}\right)}}$$

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1.219. Calculate the ratios of the following accelerations: the acceleration w_1 due to the gravitational force on the Earth's surface, the acceleration w_2 due to the centrifugal force of inertia on the Earth's equator, and the acceleration w_3 caused by the Sun to the bodies on the Earth.

Ratio:

Due to earth's gravity: $a_1 = \frac{GM_e}{R_e^2}$

1 (Quite high)

Due to centrifugal force: $a_2 = \omega^2 R_e$

0.0034 (Causes Oval shape of earth)

Due to sun's gravity: $a_3 = \frac{GM_s}{R_{\text{orbit}}^2}$

0.0006 (Still significant to cause tidal effects)

Value of g above and below earth's surface



h height above earth's surface

$$g = \frac{GM}{(R+h)^2}$$

$$= \frac{GM}{R^2 \left(1 + \frac{h}{R}\right)^2}$$

g_0

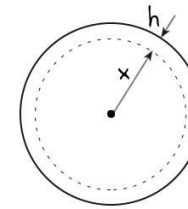
$$\approx g_0 \left(1 - \frac{2h}{R}\right) \quad (\text{approximate})$$

h height below earth's surface

We know, $g_x = \frac{GM_x}{R^3}$ $\rightarrow (R-h)$

$$= \frac{GM(R-h)}{R^3}$$

$$= g_0 \left(1 - \frac{h}{R}\right) \quad (\text{accurate})$$



Ex. value of g 2km above earth's surface is same as 1km below.

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1.220. At what height over the Earth's pole the free-fall acceleration decreases by one per cent; by half?



At h height above earth's surface,

$$g = \frac{GM}{(R+h)^2} = \frac{g_0}{\left(1 + \frac{h}{R}\right)^2} \approx g_0 \left(1 - \frac{2h}{R}\right)$$

g decreases by half

g decreases by 1%

$$\frac{g_0}{2} = \frac{g_0}{\left(1 + \frac{h}{R}\right)^2}$$

$$h = (\sqrt{2} - 1)R$$

$$= 2650 \text{ km}$$

$$g_0 \left(1 - \frac{1}{100}\right) = g_0 \left(1 - \frac{2h}{R}\right)$$

$$h = \frac{R}{200}$$

$$= 32 \text{ km}$$

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1.221. On the pole of the Earth a body is imparted velocity v_0 directed vertically up. Knowing the radius of the Earth and the free-fall acceleration on its surface, find the height to which the body will ascend. The air drag is to be neglected.

Energy conservation:

$$\frac{1}{2} m v_0^2 - \frac{G M m}{R} = - \frac{G M m}{R+h}$$

$$\frac{G M}{R^2} \rightarrow g$$

$$\Rightarrow \frac{1}{2} v_0^2 - g R = - \frac{g R}{(1 + \frac{h}{R})}$$

$$\Rightarrow h = \frac{R}{\frac{2 g R}{v_0^2} - 1} \approx \frac{v_0^2}{2 g} \text{ (if } v_0 \text{ is small)}$$

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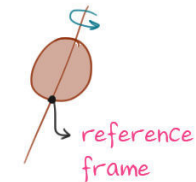
1.222. An artificial satellite is launched into a circular orbit around the Earth with velocity v relative to the reference frame moving translationally and fixed to the Earth's rotation axis. Find the distance from the satellite to the Earth's surface. The radius of the Earth and the free-fall acceleration on its surface are supposed to be known.

$$F = m a$$

$$\frac{G M m}{(R+h)^2} = \frac{m v^2}{(R+h)}$$

$$\Rightarrow h = \frac{G M}{v^2} - R \quad g = \frac{G M}{R^2}$$

$$= R \left(\frac{g R}{v^2} - 1 \right)$$



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1.223. Calculate the radius of the circular orbit of a stationary Earth's satellite, which remains motionless with respect to its surface. What are its velocity and acceleration in the inertial reference frame fixed at a given moment to the centre of the Earth?

Geostationary satellite $\omega_{\text{satellite}} = \omega_{\text{earth}}$

$$F = ma$$

$$\frac{GMm}{R^2} = m\omega^2 R \quad \rightarrow \quad \omega = \frac{2\pi}{T} \rightarrow 24 \text{ hours}$$

$$\Rightarrow R = \left[\frac{GM}{\left(\frac{2\pi}{T}\right)^2} \right]^{1/3} = 42000 \text{ km}$$

$$v = \omega R = \frac{2\pi}{T} \cdot R = 3 \text{ km/s}$$

$$a = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = 0.223 \text{ m/s}^2$$

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1.224. A satellite revolving in a circular equatorial orbit of radius $R = 2.00 \cdot 10^4 \text{ km}$ from west to east appears over a certain point at the equator every $\tau = 11.6 \text{ hours}$. Using these data, calculate the mass of the Earth. The gravitational constant is supposed to be known.

i.e same sense as earth's rotation

Time taken to 'catchup' $\tau = \frac{2\pi R}{v_{\text{rel}}} = \frac{2\pi}{\omega_{\text{rel}}} \quad (1)$
(refer 1.218) $\omega_{\text{rel}} \rightarrow \omega_{\text{sat}} - \omega_{\text{earth}}$

$$\omega_{\text{rel}} = \omega_{\text{sat}} - \omega_{\text{earth}} \quad (2)$$

$\omega_{\text{earth}} = \frac{2\pi}{T} \rightarrow 24 \text{ hours}$

$$\frac{GM}{R^2} = \omega_{\text{sat}}^2 R \quad (3)$$

From 1,2,3:

$$M = \frac{R^3}{G} \left(\frac{2\pi}{\tau} + \frac{2\pi}{T} \right)^2 = \frac{4\pi^2 R^3}{G \tau^2} \left(1 + \frac{T}{\tau} \right)^2 = 6.27 \times 10^{24} \text{ kg}$$

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1.226. A satellite must move in the equatorial plane of the Earth close to its surface either in the Earth's rotation direction or against it. Find how many times the kinetic energy of the satellite in the latter case exceeds that in the former case (in the reference frame fixed to the Earth).

wrt earth:

$$KE = \frac{1}{2} m (\omega_{rel} R)^2$$

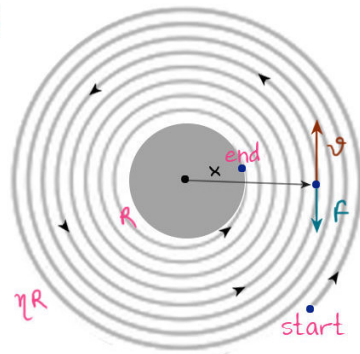
$$\therefore \frac{KE_1}{KE_2} = \frac{\omega_{rel_1}^2}{\omega_{rel_2}^2} = \left(\frac{\omega_{sat} + \omega_e}{\omega_{sat} - \omega_e} \right)^2 = \left[\frac{\sqrt{\frac{GM_e}{R_e^3}} + \frac{2\pi}{T}}{\sqrt{\frac{GM_e}{R_e^3}} - \frac{2\pi}{T}} \right]^2 \xrightarrow{\text{As its orbiting near earth's surface}} 24 \text{ hrs} \approx 1.27 //$$

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1.227. An artificial satellite of the Moon revolves in a circular orbit whose radius exceeds the radius of the Moon η times. In the process of motion the satellite experiences a slight resistance due to cosmic dust. Assuming the resistance force to depend on the velocity of the satellite as $F = \alpha v^2$, where α is a constant, find how long the satellite will stay in orbit until it falls onto the Moon's surface.

orbits are always almost circular

Method 1



$$\therefore \text{From 1,2: } \cancel{\frac{d}{dt}} \left(\frac{GMm}{2x} \right) = \cancel{\alpha v^2} v \xrightarrow{\sqrt{\frac{GM}{x}}}$$

$$\Rightarrow -\frac{GMm}{2x^2} \frac{dx}{dt} = \frac{\alpha (GM) \sqrt{GM}}{x^{3/2}}$$

$$\Rightarrow m \int_{\eta R}^R \frac{dx}{\sqrt{x}} = -2\alpha \sqrt{GM} \int_0^t dt$$

Total energy of the satellite at a given moment:

$$E = -\frac{GMm}{2x} \quad (1)$$

Rate of energy gain of satellite (Power):

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} \quad (2)$$

$$\Rightarrow t = m \sqrt{\frac{R}{GM}} \frac{(\sqrt{\eta} - 1)}{\alpha}$$

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Tangential force: $F_t = ma_t = m \frac{dv_t}{dt}$

$$\Rightarrow -\alpha v^2 = m \frac{dv}{dt}$$

$$\Rightarrow \alpha \int_0^t dt = -m \int \frac{dv}{v^2} \quad \left(\text{As the orbits are nearly circular} \right)$$

$$\alpha \int_0^t dt = -m \int \frac{dv}{v^2}$$

$$\Rightarrow t = \frac{m}{\alpha} \sqrt{\frac{R}{GM}} (\sqrt{n} - 1)$$

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1.228. Calculate the orbital and escape velocities for the Moon. Compare the results obtained with the corresponding velocities for the Earth.

At the surface

Orbital velocity: $v_o = \sqrt{\frac{GM}{R}} \quad (1)$

$$E = E_{\infty} = 0$$

Energy conservation
(As no external force)

When particle has escape velocity:
(As its KE and PE both are zero at infinity)

$$\therefore \frac{1}{2} m v_e^2 - \frac{GMm}{R} = 0 \Rightarrow v_e = \sqrt{\frac{2GM}{R}} \quad (2)$$

From 1,2

For moon:

$$v_o = 1.67 \text{ km/s}$$

$$v_e = 2.37 \text{ km/s}$$

For earth:

$$v_o = 7.9 \text{ km/s}$$

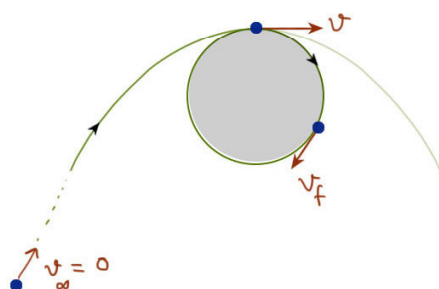
$$v_e = 11.2 \text{ km/s}$$

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1.229. A spaceship approaches the Moon along a parabolic trajectory which is almost tangent to the Moon's surface. At the moment of the maximum approach the brake rocket was fired for a short time interval, and the spaceship was transferred into a circular orbit of a Moon satellite. Find how the spaceship velocity modulus increased in the process of braking.

A particle that is at rest at infinity undergoes parabolic trajectory under gravitational force.

∴ Its total energy, $E = 0$



$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \quad \left| \quad v_f = \sqrt{\frac{GM}{R}} \right.$$

$$\Rightarrow v = \sqrt{\frac{2GM}{R}}$$

$$\Rightarrow \Delta v = v_f - v$$

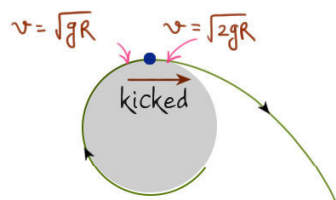
$$= -(\sqrt{2}-1)\sqrt{\frac{GM}{R}} = -(\sqrt{2}-1)\sqrt{gR}$$

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1.230. A spaceship is launched into a circular orbit close to the Earth's surface. What additional velocity has to be imparted to the spaceship to overcome the gravitational pull?

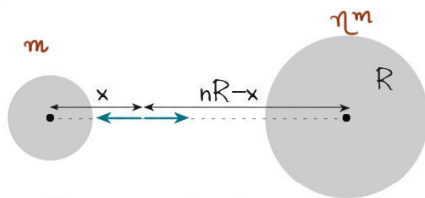
From 1.229, $\Delta v = (\sqrt{2}-1)\sqrt{gR}$

Its the minimum velocity change as its 'kicked' in same direction as its velocity.



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1.231. At what distance from the centre of the Moon is the point at which the strength of the resultant of the Earth's and Moon's gravitational fields is equal to zero? The Earth's mass is assumed to be $\eta = 81$ times that of the Moon, and the distance between the centres of these planets $n = 60$ times greater than the radius of the Earth R .



$$\frac{Gm}{x^2} = \frac{G(\eta m)}{(nR - x)^2}$$

$$\Rightarrow (nR - x)^2 = \eta x^2$$

$$\Rightarrow nR - x = \pm \sqrt{\eta} x \quad (\text{rejecting } -ve x, \text{ as gravitational force is always attractive and net force cant be zero on left side of moon})$$

$$\Rightarrow x = \frac{nR}{1 + \sqrt{\eta}}$$

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1.232. What is the minimum work that has to be performed to bring a spaceship of mass $m = 2.0 \cdot 10^3$ kg from the surface of the Earth to the Moon?

Let the net force on spaceship is zero at B. $V_B = -\frac{GM_E}{AB} - \frac{GM_M}{BC} \approx 0$ (As distances AB and BC are large)



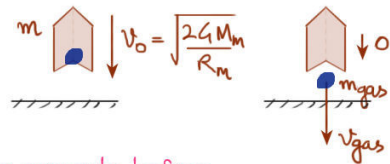
A-B Minimum work $W_{AB} = KE$ given to spaceship at A such that it just reaches point B of zero potential. i.e escape velocity.

$$\Rightarrow W_{AB} = \frac{1}{2} m v_e^2 = \frac{1}{2} m \left(\sqrt{\frac{2GM_E}{R_E}} \right)^2 = \frac{GM_E m}{R_E} \quad (1)$$

Similarly, from B to C, its velocity v_0 is again escape velocity of moon: $v_0 = \sqrt{\frac{2GM_M}{R_M}}$

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B-C Minimum work $W_{BC} = KE$ given to gases that retard the spaceship for a soft landing at moon. (A)



Two moments before hitting moon

One moment before hitting moon. (gases are released for soft landing)

momentum conservation: $mv_0 = m_g v_g$ (B) ($\because m_g < m$)

$$KE_{\text{rocket}} = \frac{1}{2} m v_0^2 = \frac{1}{2} (m v_0) v_0 = k v_0 \quad \text{(C)}$$

$$KE_{\text{gases}} = \frac{1}{2} m_g v_g^2 = \frac{1}{2} (m_g v_g) v_g = k v_g$$

Also, $v_g > v_0$ (From B and $m_g < m$) (D)

From 1,2,

Total minimum work = $W_{AB} + W_{BC}$

$$= Gm \frac{M_e}{R_e} + Gm \frac{M_m}{R_m}$$

$$= Gm \left(\frac{M_e}{R_e} + \frac{M_m}{R_m} \right)$$

From C,D:

$$\Rightarrow KE_{\text{gases}} > KE_{\text{rocket}} \quad \text{(E)}$$

From A,E:

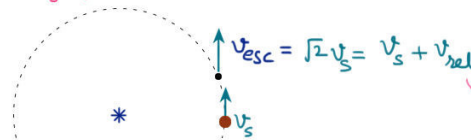
$$\Rightarrow \text{Minimum } W_{BC} = KE_{\text{rocket}}$$

$$= \frac{1}{2} m \left(\sqrt{\frac{2GM_m}{R_m}} \right)^2 = \frac{GM_m m}{R_m} \quad \text{(2)}$$

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1.233. Find approximately the third cosmic velocity v_3 , i.e. the minimum velocity that has to be imparted to a body relative to the Earth's surface to drive it out of the Solar system. The rotation of the Earth about its own axis is to be neglected.

Once it escapes earth's pull, (but is still at R_s), it should have escape velocity of sun.



$\therefore$ wrt earth, velocity needed to escape sun when it's out of earth's gravity: $v_{\text{rel}} = (\sqrt{2} - 1) v_s$

Again, wrt earth,

$$\therefore \frac{1}{2} m v_3^2 - Gm \frac{M_e}{R_e} = \frac{1}{2} m v_{\text{rel}}^2$$

$$\frac{1}{2} m v_3^2 - m v_e^2 = \frac{1}{2} m [(\sqrt{2} - 1) v_s]^2$$

$$\Rightarrow v_3^2 = 2v_e^2 + [(\sqrt{2} - 1) v_s]^2$$

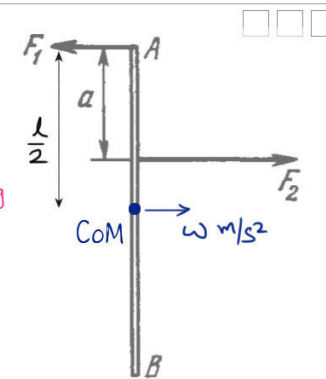
$$\Rightarrow v_3 = \sqrt{2v_e^2 + v_s^2 (\sqrt{2} - 1)^2}$$

R_s : Radius of earth's orbit around sun

R_e : Radius of earth

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1.234. A thin uniform rod AB of mass $m = 1.0 \text{ kg}$ moves translationally with acceleration $w = 2.0 \text{ m/s}^2$ due to two antiparallel forces F_1 and F_2 (Fig. 1.52). The distance between the points at which these forces are applied is equal to $a = 20 \text{ cm}$. Besides, it is known that $F_2 = 5.0 \text{ N}$. Find the length of the rod.



Not rotating

As rod is not rotating, torque about CoM must be zero.

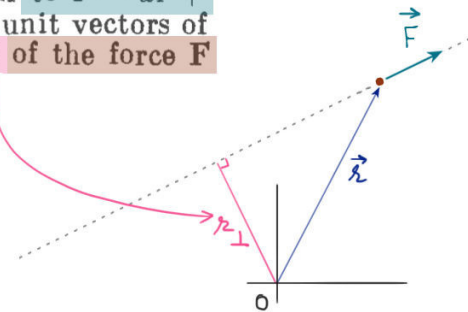
$$F_1 \frac{l}{2} = F_2 (l - a) \quad (1) \quad \Rightarrow F_2 > F_1 \quad \left(\because \frac{F_2}{F_1} = \frac{l/2}{l/2 - a} \right)$$

$$\therefore F_2 - F_1 = m w \quad (2)$$

From 1,2:
$$l = \frac{2a F_2}{m w}$$

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1.235. A force $\mathbf{F} = A\mathbf{i} + B\mathbf{j}$ is applied to a point whose radius vector relative to the origin of coordinates O is equal to $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, where a, b, A, B are constants, and $\mathbf{i}, \mathbf{j}$ are the unit vectors of the x and y axes. Find the moment N and the arm l of the force $\mathbf{F}$ relative to the point O .



$$\begin{aligned} \vec{\tau} &= \vec{r} \times \vec{F} \\ &= (a\hat{i} + b\hat{j}) \times (A\hat{i} + B\hat{j}) \\ &= (aB - bA) \hat{k} \end{aligned}$$

$$|\tau| = |\mathbf{F}| l$$

$$\Rightarrow l = \frac{|\tau|}{|\mathbf{F}|} = \frac{|aB - bA|}{\sqrt{A^2 + B^2}}$$

$$\tau = r_{\perp} F$$

arm length l

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1.236. A force $\vec{F}_1 = A\hat{j}$ is applied to a point whose radius vector $\vec{r}_1 = a\hat{i}$, while a force $\vec{F}_2 = B\hat{i}$ is applied to the point whose radius vector $\vec{r}_2 = b\hat{j}$. Both radius vectors are determined relative to the origin of coordinates O , $\hat{i}$ and $\hat{j}$ are the unit vectors of the x and y axes, a, b, A, B are constants. Find the arm l of the resultant force relative to the point O .

or $l_{\perp}$

$$\begin{aligned}\vec{\tau} &= \sum \vec{r} \times \vec{F} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 \\ &= (aA - bB)\hat{k}\end{aligned}$$

Also, $|\vec{\tau}| = l_{\perp} |\vec{F}|$

$$\Rightarrow l_{\perp} = \frac{|\vec{\tau}|}{|\vec{F}|} = \frac{|aA - bB|}{\sqrt{A^2 + B^2}}$$

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1.237. Three forces are applied to a square plate as shown in Fig. 1.53. Find the modulus, direction, and the point of application of the resultant force, if this point is taken on the side BC .

Equivalent

$$\vec{F}_{\text{net}}: \vec{F} = 2F\hat{i}$$

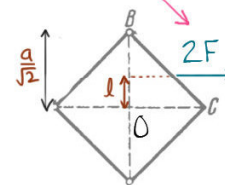
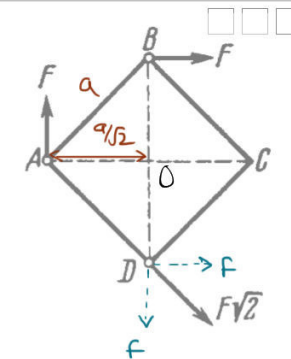
$$\text{About CoM } \vec{\tau}_{\text{net}}: \vec{\tau} = \frac{F a}{\sqrt{2}} \otimes$$

Let equivalent force $(2F\hat{i})$ on BC lie at distance l from O .
About CoM, equivalent torque should be same too.

$$\therefore 2F l = \frac{F a}{\sqrt{2}} \otimes$$

$$\Rightarrow l = \frac{a}{2\sqrt{2}}$$

$\therefore$ Equivalent force $2F\hat{i}$ will be applied at midpoint of BC .

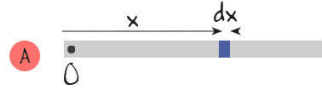


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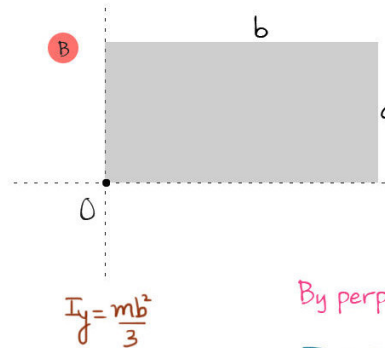
1.238. Find the moment of inertia

(a) of a thin uniform rod relative to the axis which is perpendicular to the rod and passes through its end, if the mass of the rod is m and its length l ;

(b) of a thin uniform rectangular plate relative to the axis passing perpendicular to the plane of the plate through one of its vertices, if the sides of the plate are equal to a and b , and its mass is m .



About O: $dI = dm x^2$
 $= \frac{dx}{l} m x^2$
 $I = \int dI = \frac{m}{l} \int_0^l x^2 dx$
 $= \frac{ml^2}{3}$



By perpendicular axis theorem:

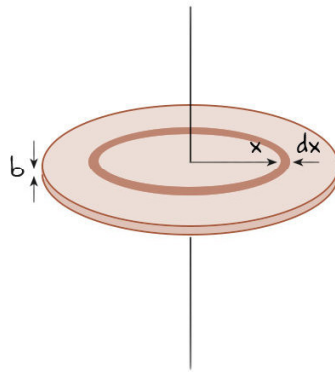
$I_0 = I_x + I_y$
 $= \frac{m}{3} (a^2 + b^2)$

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1.239. Calculate the moment of inertia

(a) of a copper uniform disc relative to the symmetry axis perpendicular to the plane of the disc, if its thickness is equal to $b = 2.0$ mm and its radius to $R = 100$ mm; Density of copper is ρ

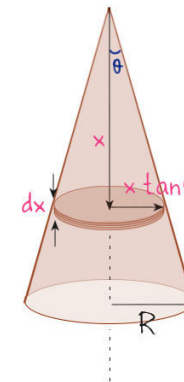
(b) of a uniform solid cone relative to its symmetry axis, if the mass of the cone is equal to m and the radius of its base to R .



A $dI = dm x^2$
 $= (\rho dV) x^2$
 $= (\rho 2\pi x dx b) x^2$
 $\Rightarrow I = 2\pi \rho b \int_0^R x^3 dx$
 $= \frac{\pi \rho b R^4}{2}$
 $= (\rho \pi R^2 b) \frac{R^2}{2}$
 $= \frac{MR^2}{2}$

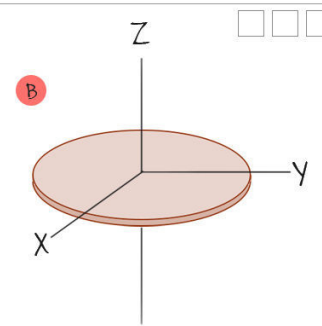
B $dI = dm \frac{(x \tan \theta)^2}{2}$ (from 1)
 $= \frac{\pi \rho dx (x \tan \theta)^4}{2}$
 $I = \frac{\pi \rho \tan^4 \theta}{2} \int_0^h x^4 dx$
 $= \frac{\pi \rho}{2} (\tan^4 \theta \frac{h^5}{5}) = \frac{\pi \rho R^4 h}{5}$
 $= (\rho \frac{1}{3} \pi R^2 h) \frac{3R^2}{10} = \frac{3mR^2}{10}$

$dm = \rho \cdot \pi (x \tan \theta)^2 dx$

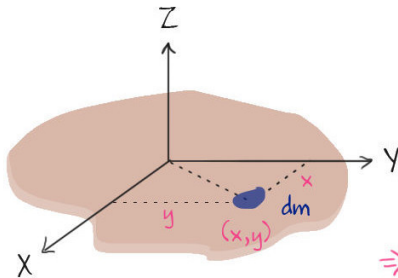


Youtube/Arihant Senior

1.240. Demonstrate that in the case of a thin plate of arbitrary shape there is the following relationship between the moments of inertia: $I_x + I_y = I_z$, where subindices x, y , and z define three mutually perpendicular axes passing through one point, with axes x and y lying in the plane of the plate. Using this relationship, find the moment of inertia of a thin uniform round disc of radius R and mass m relative to the axis coinciding with one of its diameters.



A Proof of Perpendicular axis theorem:
(Applicable to a planar object)



For a small element dm at (x, y)

$$dI_z = dm y^2$$

$$dI_y = dm x^2$$

$$dI_z = dm(x^2 + y^2)$$

$$\Rightarrow dI_x + dI_y = dI_z$$

$$\Rightarrow I_x + I_y = I_z //$$

$$I_x + I_y = I_z$$

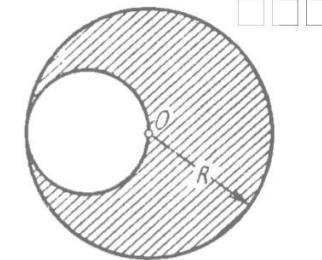
$$\Rightarrow 2I_x = I_z \text{ (by symmetry)}$$

$$\Rightarrow 2I_x = \frac{mR^2}{2}$$

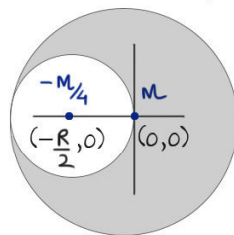
$$\Rightarrow I_x = \frac{mR^2}{4} //$$

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1.241. A uniform disc of radius $R = 20$ cm has a round cut as shown in Fig. 1.54. The mass of the remaining (shaded) portion of the disc equals $m = 7.3$ kg. Find the moment of inertia of such a disc relative to the axis passing through its centre of inertia and perpendicular to the plane of the disc.



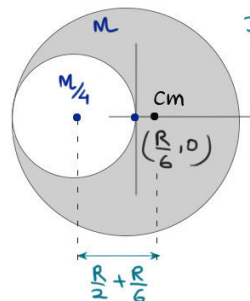
Let the mass of complete disc be M .



$$x_{cm} = \frac{M(0) + (-\frac{M}{4})(-\frac{R}{2})}{M + (-\frac{M}{4})}$$

$$= \frac{\frac{MR}{8}}{\frac{3M}{4}} = +\frac{R}{6}$$

$$m = \frac{3M}{4} \quad M \quad -\frac{M}{4}$$



$$I_{cm} = I_{M|cm} - I_{M/4|cm}$$

$$= \left[\frac{MR^2}{2} + M\left(\frac{R}{6}\right)^2 \right] - \left[\frac{(M/4)(R/2)^2}{2} + \left(\frac{M}{4}\right)\left(\frac{R}{2} + \frac{R}{6}\right)^2 \right]$$

$$= \frac{4m}{3} R^2 \left(\frac{1}{2} + \frac{1}{36} - \frac{1}{32} - \frac{1}{9} \right)$$

$$= \frac{37}{72} mR^2 //$$

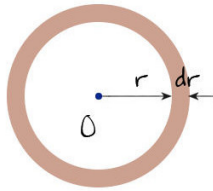
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1.242. Using the formula for the moment of inertia of a uniform sphere, find the moment of inertia of a thin spherical layer of mass m and radius R relative to the axis passing through its centre.

For a sphere:

$$I = \frac{2}{5} MR^2 = \frac{2}{5} \left[\frac{4}{3} \pi R^3 \right] R^2 = \frac{2}{5} \cdot \frac{4}{3} \pi R^5 = CR^5$$

For a spherical shell: $I_{\text{shell}} = I_{\text{Big sphere}} - I_{\text{small sphere}}$



$$\begin{aligned} &= I_{r+dr} - I_r \\ &= C(r+dr)^5 - Cr^5 \\ &= Cr^5 \left[\left(1 + \frac{dr}{r}\right)^5 - 1 \right] \\ &\approx Cr^5 \cdot \frac{5dr}{r} \end{aligned}$$

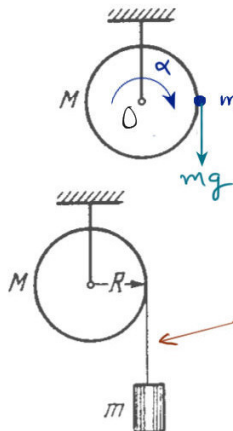
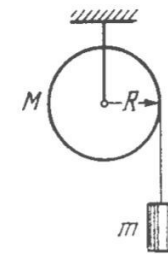
Mass of shell:

$$m = 4\pi r^2 dr \quad \Rightarrow \quad 5Cr^4 dr = 5 \left(\frac{2}{5} \cdot \frac{4}{3} \pi \right) r^4 \cdot \frac{m}{4\pi r^2} = \frac{2}{3} mr^2 //$$

Youtube / Arihant Senior

1.243. A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass M and radius R (Fig. 1.55). At a moment $t = 0$ the system is set in motion. Assuming the friction in the axle of the cylinder to be negligible, find the time dependence of

- the angular velocity of the cylinder;
- the kinetic energy of the whole system.



Stick m here to get α as T is now internal force.

Then use α in original situation to get values of Tension, acceleration of m etc.

Sol: $\tau_0 = I_0 \alpha$

$$\Rightarrow mgr = \left(\frac{MR^2}{2} + mR^2 \right) \alpha$$

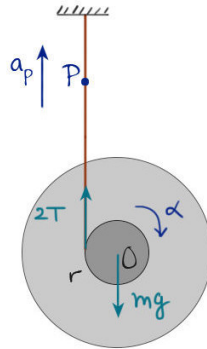
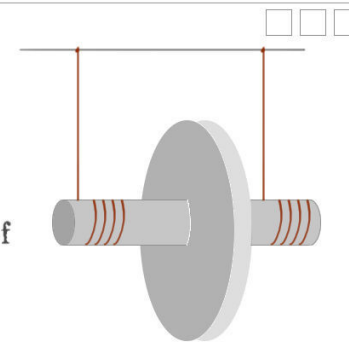
$$\Rightarrow \alpha = \frac{mgr}{I_0}$$

A $\omega = \alpha t = \frac{mgrt}{I_0} //$

B $KE = \frac{1}{2} I_0 \omega^2 = \frac{1}{2} I_0 \cdot \frac{m^2 g^2 R^2}{I_0^2} t^2 = \frac{1}{2} \frac{m^2 g^2 t^2}{\left(\frac{M}{2} + m \right)} //$

Youtube / Arihant Senior

1.244. The ends of thin threads tightly wound on the axle of radius r of the Maxwell disc are attached to a horizontal bar. When the disc unwinds, the bar is raised to keep the disc at the same height. The mass of the disc with the axle is equal to m , the moment of inertia of the arrangement relative to its axis is I . Find the tension of each thread and the acceleration of the bar.



∴ The disc is kept at same height,

A $2T = mg$

B $a_p = \alpha r$
 $= \frac{mgr^2}{I}$

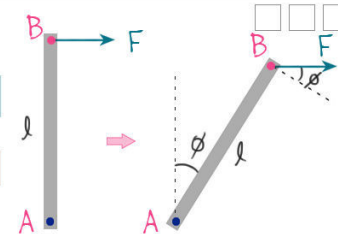
Torque about O

$I\alpha = 2Tr$

$\Rightarrow \alpha = \frac{mgr}{I}$

Youtube / Arihant Senior

1.245. A thin horizontal uniform rod AB of mass m and length l can rotate freely about a vertical axis passing through its end A . At a certain moment the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in a horizontal plane. Find the angular velocity of the rod as a function of its rotation angle ϕ counted relative to the initial position.



Work = $\vec{r} \cdot \vec{F} = \Delta KE = \frac{1}{2} I \omega^2$

Displacement of point of application of force (i.e. point B)

$\Rightarrow F(l \sin \phi) = \frac{1}{2} \frac{ml^2}{3} \omega^2 \Rightarrow \omega = \sqrt{\frac{6F \sin \phi}{ml}}$

Method 1

Method 2 $\tau = I\alpha$

$\Rightarrow F \cos \phi l = \frac{ml^2}{3} \cdot \frac{\omega d\omega}{d\phi}$

$\Rightarrow Fl \int_0^\phi \cos \phi d\phi = \frac{ml^2}{3} \int_0^\omega \omega d\omega$

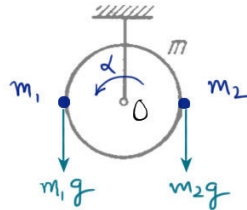
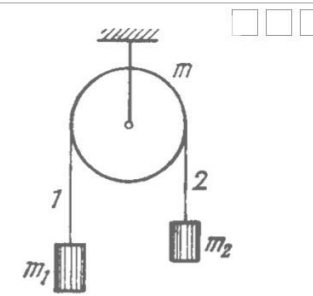
Method 3

$\tau d\theta = dKE$

$\int_0^\phi Fl \cos \phi d\phi = \int_0^\omega \frac{1}{2} I \omega^2 d\omega$

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1.246. In the arrangement shown in Fig. 1.56 the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2 . The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions T_1/T_2 of the vertical sections of the thread in the process of motion.



Stick m_1 and m_2 to get α as T is now internal Force.

Then use it in original situation to get values of Tension, acceleration of masses etc.

Sol: $\tau = I_0 \alpha$

$$\Rightarrow (m_1 g - m_2 g) R = \left(\frac{m R^2}{2} + m_1 R^2 + m_2 R^2 \right) \alpha$$

$$\Rightarrow \alpha = \frac{(m_1 - m_2) g}{\left(\frac{m}{2} + m_1 + m_2 \right) R}$$

$$m_1 g - T_1 = m_1 (\alpha R)$$

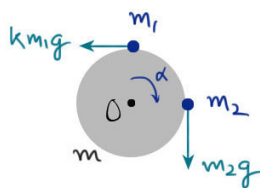
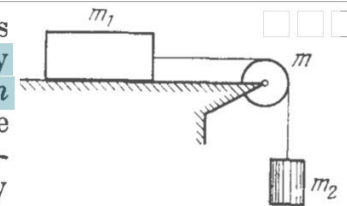
$$T_2 - m_2 g = m_2 (\alpha R)$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{m_1 (g - \alpha R)}{m_2 (g + \alpha R)}$$

$$= \frac{m_1 (m + 4m_2)}{m_2 (m + 4m_1)}$$

Youtube / Arihant Senior

1.247. In the system shown in Fig. 1.57 the masses of the bodies are known to be m_1 and m_2 , the coefficient of friction between the body m_1 and the horizontal plane is equal to k , and a pulley of mass m is assumed to be a uniform disc. The thread does not slip over the pulley. At the moment $t = 0$ the body m_2 starts descending. Assuming the mass of the thread and the friction in the axle of the pulley to be negligible, find the work performed by the friction forces acting on the body m_1 over the first t seconds after the beginning of motion.



Stick m_1 and m_2 to get α as T is now internal Force.

$$\tau = I_0 \alpha$$

$$(m_2 g - k m_1 g) R = \left(\frac{m R^2}{2} + m_1 R^2 + m_2 R^2 \right) \alpha$$

$$\Rightarrow \alpha = \frac{(m_2 - k m_1) g}{\left(\frac{m}{2} + m_1 + m_2 \right) R}$$

$$s = \frac{1}{2} a t^2$$

$$f = k m_1 g$$

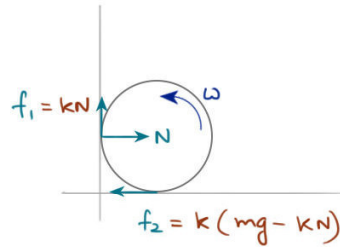
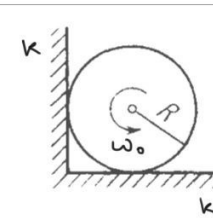
$$W_f = \vec{f} \cdot \vec{s}$$

$$= -(k m_1 g) \cdot \frac{1}{2} (\alpha R) t^2$$

$$= -\frac{k m_1 g^2 (m_2 - k m_1) t^2}{m + 2(m_1 + m_2)}$$

Youtube / Arihant Senior

1.248. A uniform cylinder of radius R is spun about its axis to the angular velocity ω_0 and then placed into a corner (Fig. 1.58). The coefficient of friction between the corner walls and the cylinder is equal to k . How many turns will the cylinder accomplish before it stops?



$$F_x = 0 \Rightarrow N = k(mg - kN) \quad (1)$$

$$\tau = I\alpha \Rightarrow [kN + k(mg - kN)]R = \frac{mR^2}{2}\alpha \quad (2)$$

$$\text{From 1,2: } \alpha = \frac{2kg(1+k)}{R(1+k^2)}$$

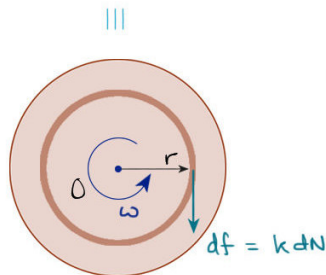
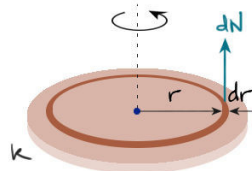
When it stops,

$$0 = \omega_0^2 - 2\alpha\theta \Rightarrow \theta = \frac{\omega_0^2}{2\alpha}$$

$$\therefore \text{No. of revolutions: } \frac{\theta}{2\pi} = \frac{\omega_0^2}{4\pi\alpha} = \frac{\omega_0^2 R(1+k^2)}{8\pi kg(1+k)} //$$

Youtube / Arihant Senior

1.249. A uniform disc of radius R is spinned to the angular velocity ω_0 and then carefully placed on a horizontal surface. How long will the disc be rotating on the surface if the friction coefficient is equal to k ? The pressure exerted by the disc on the surface can be regarded as uniform.



Torque on elemental ring on the disc is:

$$\begin{aligned} dz &= k dN r \\ &= k(dm g) r \\ &= k \left(\frac{2\pi r dr}{\pi R^2} m g \right) r \end{aligned}$$

$$\begin{aligned} \Rightarrow \tau &= \int_0^R dz = \frac{2kmg}{R^2} \int_0^R r^2 dr \\ &= \frac{2kmgR}{3} \end{aligned}$$

On the whole disc:

$$\begin{aligned} \tau &= I\alpha \\ \Rightarrow \alpha &= \frac{\tau}{I} = \frac{2kmgR}{3 \frac{mR^2}{2}} = \frac{4kg}{3R} \end{aligned}$$

When disc stops:

$$\begin{aligned} 0 &= \omega_0 - \alpha t \\ \Rightarrow t &= \frac{\omega_0}{\alpha} = \frac{3\omega_0 R}{4kg} // \end{aligned}$$

Youtube / Arihant Senior

1.250. A flywheel with the initial angular velocity ω_0 decelerates due to the forces whose moment relative to the axis is proportional to the square root of its angular velocity. Find the mean angular velocity of the flywheel averaged over the total deceleration time.

As its decelerating

$$\tau \propto -\sqrt{\omega}$$

$$\Rightarrow \tau = -C\sqrt{\omega}$$

$$\Rightarrow I\omega \frac{d\omega}{d\theta} = -C\sqrt{\omega}$$

$$\Rightarrow I \int_{\omega_0}^0 \sqrt{\omega} d\omega = -C \int_0^{\theta} d\theta$$

$$\Rightarrow \theta = \frac{2}{3} \frac{I}{C} \omega_0^{3/2} \quad (1)$$

$$I \frac{d\omega}{dt} = -C\sqrt{\omega}$$

$$\Rightarrow I \int_{\omega_0}^0 \frac{d\omega}{\sqrt{\omega}} = -C \int_0^t dt$$

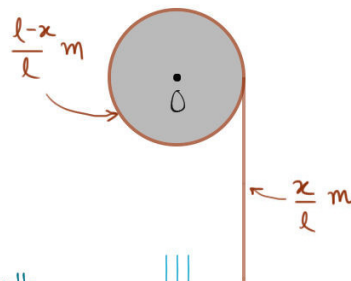
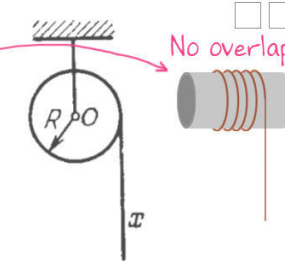
$$\Rightarrow t = \frac{2I\sqrt{\omega_0}}{C} \quad (2)$$

From 1,2

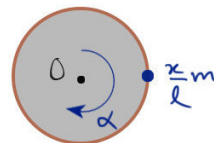
$$\omega_{av} = \frac{\theta}{t} = \frac{\omega_0}{3} //$$

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1.251. A uniform cylinder of radius R and mass M can rotate freely about a stationary horizontal axis O (Fig. 1.59). A thin cord of length l and mass m is wound on the cylinder in a single layer. Find the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. The wound part of the cord is supposed to have its centre of gravity on the cylinder axis.



Stick hanging mass on pulley to get α as T is now internal Force.



$$\tau_O = I_O \alpha$$

$$\frac{x}{l} mgR = \left(\frac{MR^2}{2} + \frac{l-x}{l} mR^2 + \frac{x}{l} mR^2 \right) \alpha$$

$$\Rightarrow \frac{x}{l} mgR = \left(\frac{MR^2}{2} + mR^2 \right) \alpha$$

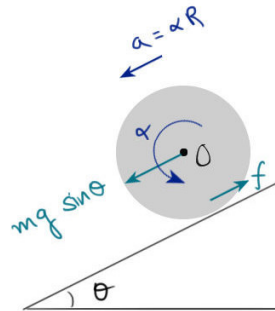
$$\Rightarrow \alpha = \frac{mgx}{lR(\frac{M}{2} + m)} //$$

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1.252. A uniform sphere of mass m and radius R rolls without slipping down an inclined plane set at an angle θ to the horizontal.

Find:

- (a) the magnitudes of the friction coefficient at which slipping is absent;
(b) the kinetic energy of the sphere t seconds after the beginning of motion.



A When its rolling without slipping,

Torque: $fR = I_0 \alpha$ (1)

Force: $mg \sin \theta - f = m(\alpha R)$ (2)

From 1,2:

$$f = \frac{mg \sin \theta}{1 + \frac{mR^2}{I_0}} \leq \mu mg \cos \theta$$

$\frac{2}{5}mR^2$

$$\Rightarrow \mu \geq \frac{2}{7} \tan \theta //$$

If $I_{cm} = k m R^2$

$$KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$= \frac{1}{2} m v^2 (1 + k)$$

B From 1,2:

$$\alpha = \frac{mg \sin \theta R}{I_0 + mR^2}$$

$$KE = \frac{1}{2} m (\alpha R t)^2 (1 + \frac{2}{5})$$

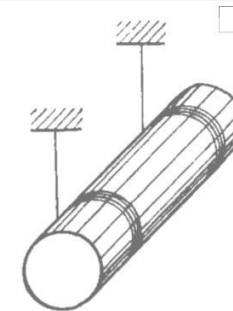
$$= \frac{1}{2} m \frac{m^2 g^2 \sin^2 \theta R^2 t^2}{m^2 R^4 (\frac{7}{5})^2}$$

$$= \frac{5}{14} m g^2 \sin^2 \theta t^2$$

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1.253. A uniform cylinder of mass $m = 8.0$ kg and radius $R = 1.3$ cm (Fig. 1.60) starts descending at a moment $t = 0$ due to gravity. Neglecting the mass of the thread, find:

- (a) the tension of each thread and the angular acceleration of the cylinder;
(b) the time dependence of the instantaneous power developed by the gravitational force.



A Torque: $\tau_p = I_p \alpha$

$$\Rightarrow mgR = (\frac{mR^2}{2} + mR^2) \alpha$$

$$\Rightarrow \alpha = \frac{2g}{3R} //$$

Force:

$$mg - 2T = m(\alpha R)$$

$$\Rightarrow T = \frac{m}{2} (g - \alpha R)$$

$$= \frac{mg}{6} //$$

B $P = Fv$

$\therefore$ Power due to gravity:

$$P_g = mg \cdot v$$

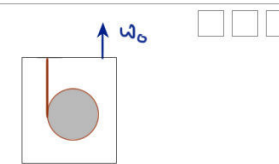
$$= mg(\alpha R t)$$

$$= mg \cdot \frac{2g}{3R} \cdot R t$$

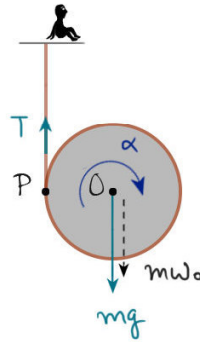
$$= \frac{2}{3} m g^2 t //$$

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1.254. Thin threads are tightly wound on the ends of a uniform solid cylinder of mass m . The free ends of the threads are attached to the ceiling of an elevator car. The car starts going up with an acceleration w_0 . Find the acceleration w' of the cylinder relative to the car and the force F exerted by the cylinder on the ceiling (through the threads).



w.r.t elevator:



Torque about P:

$$\tau_P = I_P \alpha$$

$$\Rightarrow (mg + mw_0)R = \left(\frac{mR^2}{2} + mR^2\right) \alpha$$

$$\Rightarrow \alpha = \frac{2(g + w_0)}{3R}$$

$$\therefore w' = \alpha R = \frac{2(g + w_0)}{3}$$

Torque about O:

$$\tau_O = I_O \alpha$$

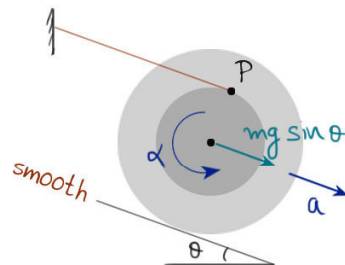
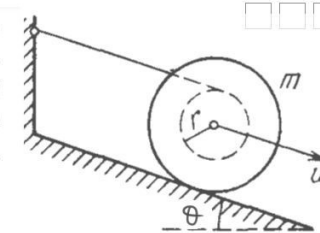
$$\Rightarrow T R = \frac{mR^2}{2} \cdot \frac{2(g + w_0)}{3R}$$

$$\Rightarrow T = \frac{m(g + w_0)}{3}$$

Note: θ, ω, α of a rotating body is independent of translational frame of reference. Be it accelerating or non-accelerating.

Youtube / Arihant Senior

1.255. A spool with a thread wound on it is placed on an inclined smooth plane set at an angle $\theta = 30^\circ$ to the horizontal. The free end of the thread is attached to the wall as shown in Fig. 1.61. The mass of the spool is $m = 200$ g, its moment of inertia relative to its own axis $I = 0.45$ g·m², the radius of the wound thread layer $r = 3.0$ cm. Find the acceleration of the spool axis.



$$\text{Torque: } \tau_P = I_P \alpha$$

$$\Rightarrow mg \sin \theta r = (I + mr^2) \alpha$$

$$\Rightarrow \alpha = \frac{mg \sin \theta r}{I + mr^2}$$

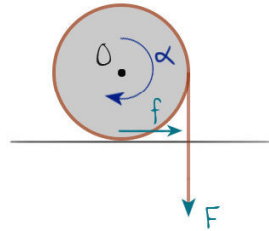
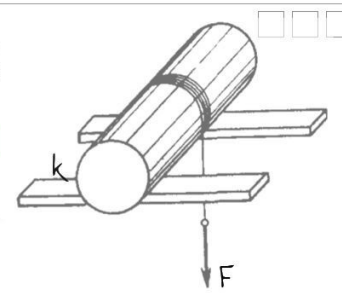
$\therefore P$ is at rest, wrt P ,

$a = \alpha r$ (Even though there is slipping!!! Besides, it's r , NOT R)

$$= \frac{mg \sin \theta r^2}{I + mr^2}$$

Youtube / Arihant Senior

1.256. A uniform solid cylinder of mass m rests on two horizontal planks. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically down with a constant force F (Fig. 1.62). Find the maximum magnitude of the force F which still does not bring about any sliding of the cylinder, if the coefficient of friction between the cylinder and the planks is equal to k . What is the acceleration a_{max} of the axis of the cylinder rolling



Torque: $\tau_o = I_o \alpha$

$$\Rightarrow (F - f)R = \frac{mR^2}{2} \alpha \quad (1)$$

Force: $f = m(\alpha R) \quad (2)$

From 1,2:

$$f = \frac{2F}{3} \leq k(mg + F)$$

$$\Rightarrow F \leq \frac{3kmg}{2-3k} //$$

From 1,2:

$$\alpha = \frac{2F}{3mR}$$

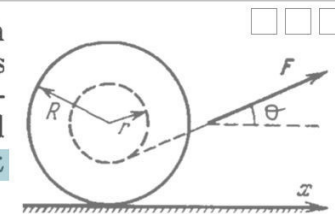
$$a_{max} = \alpha_{max} R$$

$$= \left(\frac{2F_{max}}{3mR} \right) R$$

$$= \frac{2 \cdot \frac{3kmg}{2-3k}}{3m} = \frac{2kg}{2-3k} //$$

Youtube / Arihant Senior

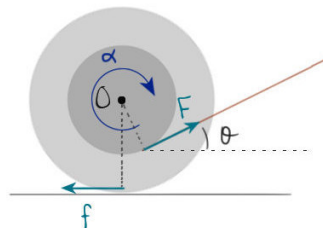
1.257. A spool with thread wound on it, of mass m , rests on a rough horizontal surface. Its moment of inertia relative to its own axis is equal to $I = \gamma mR^2$, where γ is a numerical factor, and R is the outside radius of the spool. The radius of the wound thread layer is equal to r . The spool is pulled without sliding by the thread with a constant force F directed at an angle θ to the horizontal (Fig. 1.63). Find:



(a) the projection of the acceleration vector of the spool axis on the x -axis;

(b) the work performed by the force F during the first t seconds after the beginning of motion.

Method 1



A Torque: $\tau_o = I_o \alpha$

$$\Rightarrow fR - F r \sin \theta = \gamma mR^2 \alpha \quad (1)$$

Force: $F \cos \theta - f = m(\alpha R) \quad (2)$

$$\text{From 1,2 } \alpha = \frac{F[R \cos \theta - r]}{mR^2(1+\gamma)}$$

$$\therefore a = \alpha R //$$

B As work done by friction = 0,

$$W = \Delta KE$$

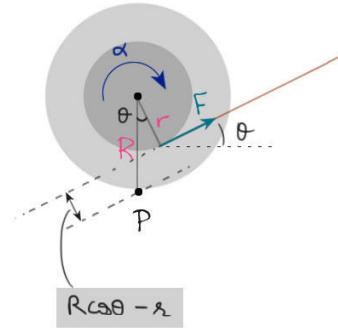
$$= \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m \omega^2 R^2 + \frac{1}{2} \gamma m r^2 \omega^2$$

$$= \frac{1}{2} m R^2 (1+\gamma) \alpha^2 t^2 //$$

Youtube / Arihant Senior

Method 2



$$\tau_P = I_P \alpha$$

$$\Rightarrow F(R \cos \theta - r) = (I + MR^2) \alpha$$

$$\Rightarrow \alpha = \frac{F(R \cos \theta - r)}{MR^2(1 + 1)} //$$

∴ Three cases:

$$1 \quad \cos \theta > \frac{r}{R}, \quad \alpha > 0$$

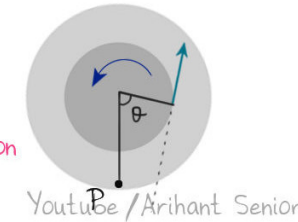
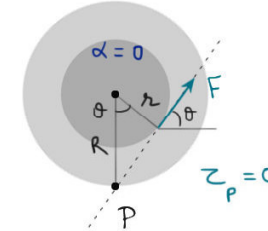
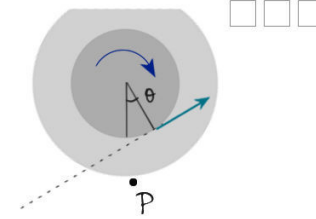
It will roll in +ve X direction

$$2 \quad \cos \theta = \frac{r}{R}, \quad \alpha = 0$$

(F passes through P then.)

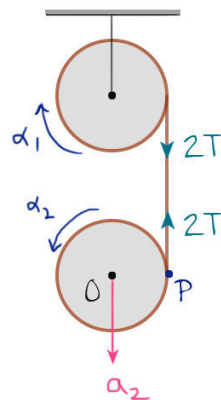
$$3 \quad \cos \theta < \frac{r}{R}, \quad \alpha < 0$$

Then it will roll in -ve X direction



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1.258. The arrangement shown in Fig. 1.64 consists of two identical uniform solid cylinders, each of mass m , on which two light threads are wound symmetrically. Find the tension of each thread in the process of motion. The friction in the axle of the upper cylinder is assumed to be absent.



Torque on each cylinder:

$$2TR = I\alpha_1$$

$$2TR = I\alpha_2$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha = \frac{2TR}{I}$$

$$\text{Also, } a_2 = a_P + \alpha R$$

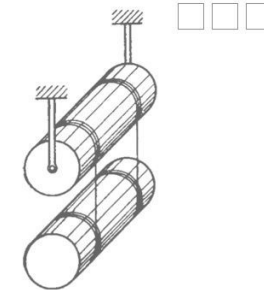
$$= 2\alpha R$$

$$= 2 \cdot \frac{2TR}{I} \cdot R$$

$$= \frac{8T}{m} \quad \left(\text{since } I = \frac{mR^2}{2} \right)$$

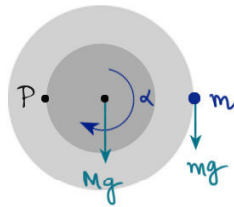
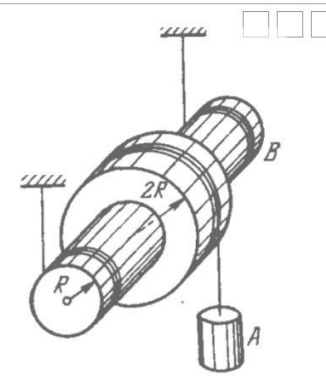
$$\text{Force on 2: } mg - 2T = ma_2$$

$$\Rightarrow T = \frac{mg}{10}$$



Youtube / Arihant Senior

1.259. In the arrangement shown in Fig. 1.65 a weight A possesses mass m , a pulley B possesses mass M . Also known are the moment of inertia I of the pulley relative to its axis and the radii of the pulley R and $2R$. The mass of the threads is negligible. Find the acceleration of the weight A after the system is set free.



Stick m to get α
as T is now internal Force.

Sol: $\sum \tau = I_P \alpha$

$$MgR + mg3R = [(I + MR^2) + m(3R)^2] \alpha$$

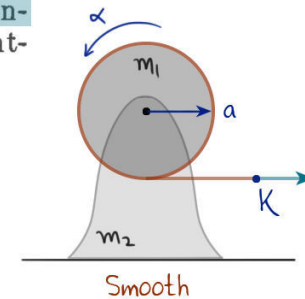
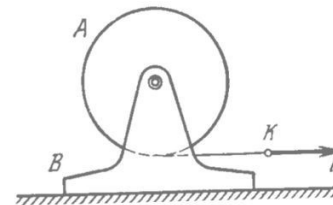
$$\Rightarrow \alpha = \frac{gR(M + 3m)}{I + MR^2 + 9mR^2}$$

$$\therefore a_A = \alpha \cdot 3R //$$

Youtube / Arihant Senior

1.260. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount B of mass m_2 (Fig. 1.66). A constant horizontal force F is applied to the end K of a light thread tightly wound on the cylinder. The friction between the mount and the supporting horizontal plane is assumed to be absent. Find:

- the acceleration of the point K ;
- the kinetic energy of this system t seconds after the beginning of motion.



Torque: $FR = I\alpha$

Force: $F = (m_1 + m_2)a$

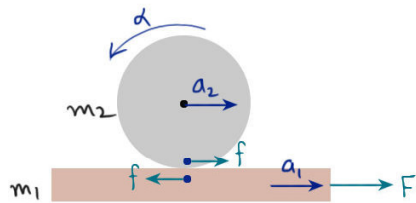
A $a_K = a + \alpha R$

$$= \frac{F}{m_1 + m_2} + \frac{FR^2}{I} = F \left[\frac{1}{m_1 + m_2} + \frac{2}{m_1} \right] //$$

B $KE = W_F$
 $= \vec{F} \cdot \vec{s}$
 Displacement of K
 $= F \left(\frac{1}{2} a_K t^2 \right) //$

Youtube / Arihant Senior

1.261. A plank of mass m_1 with a uniform sphere of mass m_2 placed on it rests on a smooth horizontal plane. A constant horizontal force F is applied to the plank. With what accelerations will the plank and the centre of the sphere move provided there is no sliding between the plank and the sphere?



Forces on both:

$$F - f = m_1 a_1 \quad (1)$$

$$f = m_2 a_2 \quad (2)$$

$$\text{Torque: } fR = I\alpha \quad (3)$$

$$I = \frac{2}{5}m_2 R^2$$

$$\text{No slipping: } a_2 + \alpha R = a_1 \quad (4)$$

From 1,2,3,4:

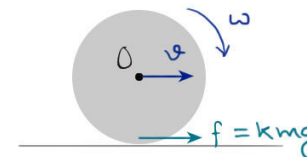
$$a_1 = \frac{F}{m_1 + \frac{2}{7}m_2}, \quad a_2 = \frac{2}{7}a_1 //$$

Youtube/Arihant Senior

1.262. A uniform solid cylinder of mass m and radius R is set in rotation about its axis with an angular velocity ω_0 , then lowered with its lateral surface onto a horizontal plane and released. The coefficient of friction between the cylinder and the plane is equal to k . Find:

- (a) how long the cylinder will move with sliding;
(b) the total work performed by the sliding friction force acting on the cylinder.

At a general time t :



$$\text{Torque: } \tau_0 = I_0 \alpha$$

$$\Rightarrow kmgR = \frac{mR^2}{2} \alpha$$

$$\Rightarrow \alpha = \frac{2kg}{R}$$

$$\text{Force: } f = ma$$

$$\Rightarrow kmg = ma$$

$$\Rightarrow a = kg //$$

A When rolling begins:

$$v = \omega R$$

$$\Rightarrow 0 + at = (\omega_0 - \alpha t)R$$

$$\Rightarrow kgt = \left(\omega_0 - \frac{2kg}{R}t\right)R$$

$$\Rightarrow t = \frac{\omega_0 R}{3kg} //$$

B $W_f = \Delta KE$

$$= \left[\frac{1}{2} m v^2 + \frac{1}{2} I \left(\frac{v}{R} \right)^2 \right] - \frac{1}{2} I \omega_0^2$$

$$= \frac{1}{2} \cdot \frac{3}{2} m v^2 - \frac{m R^2}{4} \omega_0^2$$

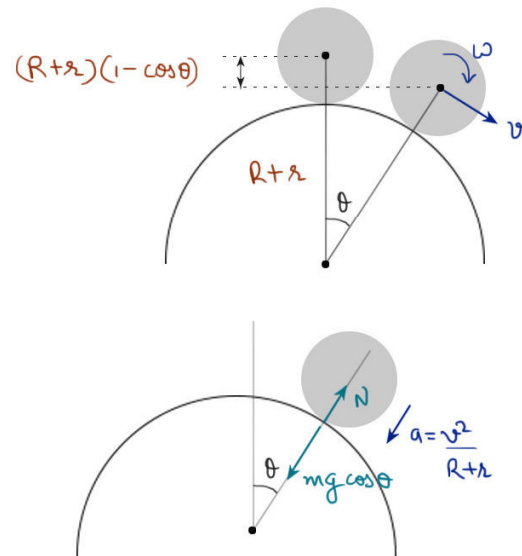
$$at = \frac{\omega_0 R}{3}$$

$$= \frac{3}{4} m \frac{\omega_0^2 R^2}{3^2} - \frac{m \omega_0^2 R^2}{4}$$

$$= - \frac{m \omega_0^2 R^2}{4}$$

Youtube/Arihant Senior

1.263. A uniform ball of radius r rolls without slipping down from the top of a sphere of radius R . Find the angular velocity of the ball at the moment it breaks off the sphere. The initial velocity of the ball is negligible.



Energy conservation:

$$mg(R+r)(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$$

$$\Rightarrow v^2 = \frac{10}{7}g(R+r)(1-\cos\theta) \quad (1)$$

Force along normal surface: $F = ma$

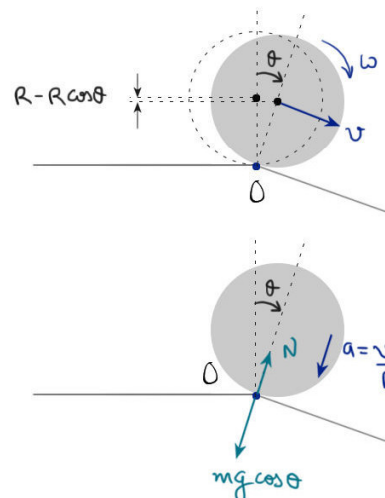
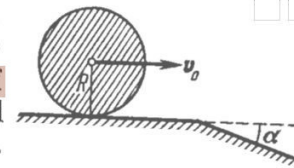
$$\Rightarrow mg\cos\theta - N = \frac{mv^2}{R+r} \quad (2)$$

(when it breaks off)

From 1,2: $v = \sqrt{\frac{10}{17}g(R+r)}$, $\omega = \frac{v}{r}$

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1.264. A uniform solid cylinder of radius $R = 15$ cm rolls over a horizontal plane passing into an inclined plane forming an angle $\alpha = 30^\circ$ with the horizontal (Fig. 1.67). Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.



Cylinder will roll with Instantaneous axis of rotation O.

Energy convs: $W_g = \Delta KE$

$$\Rightarrow mgr(1-\cos\theta) = \frac{1}{2}I_O\left(\frac{v}{R}\right)^2 - \frac{1}{2}I_O\left(\frac{v_0}{R}\right)^2 \quad (1)$$

$$\text{Force: } mg\cos\theta - N = \frac{mv^2}{R}$$

$$\Rightarrow N = mg\cos\theta - \frac{mv^2}{R} \quad (2)$$

N decreases as v and θ increase.

So N is minimum at maximum θ , i.e. α

$\therefore$ When N just becomes zero when $\theta = \alpha$, we get the maximum value v_0 at which it won't jump.

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Solving 1 and 2 with $N=0$ and $\theta = \alpha$

Eliminating v^2

$$mgR(1 - \cos\theta) = \frac{1}{2} \frac{I_0}{R^2} (g\cos\theta \cdot R - v_0^2)$$

$\frac{3}{2}mR^2$

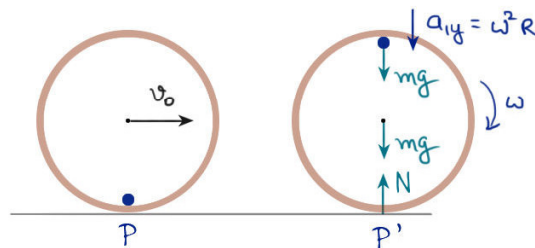
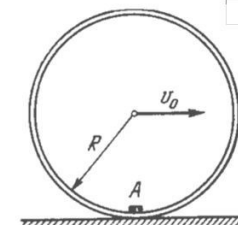
$$\Rightarrow v_0 = \sqrt{\frac{(7\cos\theta - 4)gR}{3}}$$

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1.265. A small body A is fixed to the inside of a thin rigid hoop of radius R and mass equal to that of the body A . The hoop rolls without slipping over a horizontal plane; at the moments when the body A gets into the lower position, the centre of the hoop moves with velocity v_0 (Fig. 1.68). At what values of v_0 will the hoop move without bouncing?

Normal reaction is minimum when A reaches top (Proof in next slide)

$\therefore$ If N becomes zero, just when A reaches top, we get maximum v_0



Force in Y direction for the system:

$$F_y = m_1 a_{1y} + m_2 a_{2y} \quad \text{(As hoop travels along X only)}$$

$$\Rightarrow 2mg - N = m\omega^2 R \quad (1)$$

Energy: $mg(2R) = \frac{1}{2} I_P \left(\frac{v_0}{R}\right)^2 - \frac{1}{2} I_{P'} \omega^2$

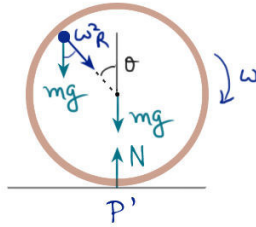
$$\Rightarrow 2mgR = \frac{1}{2} (2mR^2) \left(\frac{v_0}{R}\right)^2 - \frac{1}{2} (2mR^2 + m(2R)^2) \omega^2 \quad (2)$$

From 1,2:

$$v_0 = \sqrt{8gR}$$

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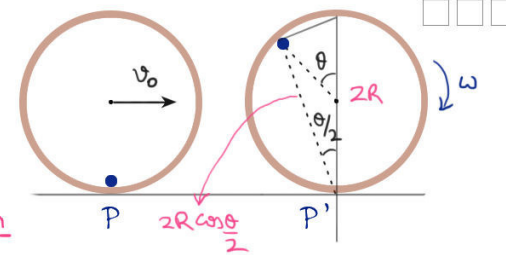
$$a_{iy} = \omega^2 R \cos \theta$$



$$2mg - N = m\omega^2 R \cos \theta$$

$$\Rightarrow N = 2mg - m\omega^2 R \cos \theta$$

∴ For minimum N , $\omega^2 \cos \theta$ should be maximum



Energy conservation:

$$mgR(1 + \cos \theta) = \frac{1}{2} I_P \left(\frac{v_0}{R} \right)^2 - \frac{1}{2} I_{P'} \omega^2$$

$$\Rightarrow mgR(1 + \cos \theta) = \frac{1}{2} (2mR^2) \left(\frac{v_0}{R} \right)^2 - \frac{1}{2} (2mR^2 + m(2R \cos \frac{\theta}{2})^2) \omega^2$$

$$\Rightarrow v_0^2 - gR(1 + \cos \theta) = (1 + 2 \cos^2 \frac{\theta}{2}) R^2 \omega^2$$

$$\Rightarrow \omega^2 = \frac{v_0^2 - gR(1 + \cos \theta)}{R^2(2 + \cos \theta)}$$

$$\therefore \omega^2 \cos \theta = \frac{v_0^2 - gR(1 + \cos \theta)}{R^2(2 + \cos \theta)} \cos \theta$$

$f(\theta)$

$$f'(\theta) = 0 \text{ at } \sin \theta = 0$$

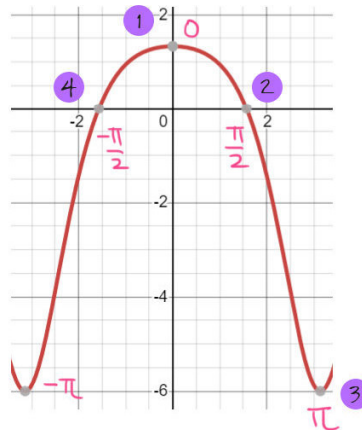
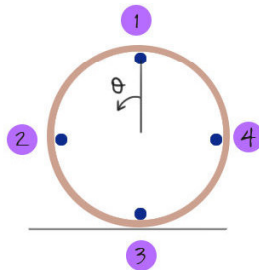
$$\Rightarrow \theta = 0$$

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$$N = 2mg - m\omega^2 R \cos \theta$$

For minimum N , $\omega^2 \cos \theta$ should be maximum

$$\therefore \omega^2 \cos \theta = f(\theta) = \frac{v_0^2 - gR(1 + \cos \theta)}{R^2(2 + \cos \theta)} \cos \theta$$



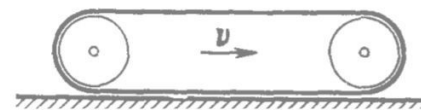
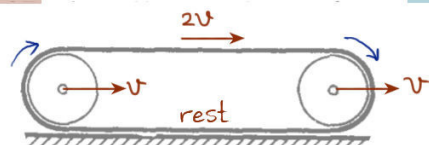
A N is same when mass is climbing up and when its climbing down at symmetric locations.

B At bottom half, i.e. $-ve \cos \theta$
 N is always positive, regardless of v_0

∴ It will never break contact there.

Youtube / Arihant Senior

1.266. Determine the kinetic energy of a tractor crawler belt of mass m if the tractor moves with velocity v (Fig. 1.69).



reconfigure in three parts

1 $KE_1 = m_A v^2$

2 $KE_2 = \frac{1}{2} m_B (2v)^2$

3 $KE_3 = 0$

When a ring rolls, its $KE = \frac{1}{2} m v^2 + \frac{1}{2} I \left(\frac{v}{r} \right)^2 = m v^2$ $\rightarrow m r^2$

Total $KE = KE_1 + KE_2 + KE_3$

$$= m_A v^2 + 2 m_B v^2 + 0$$

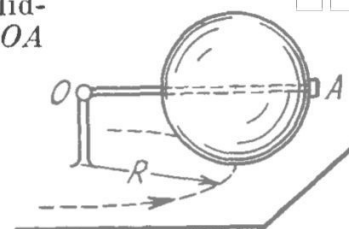
$$= (m_A + 2 m_B) v^2$$

$$= m v^2 //$$

(Independent of radius of rollers)

Youtube / Arihant Senior

1.267. A uniform sphere of mass m and radius r rolls without sliding over a horizontal plane, rotating about a horizontal axle OA (Fig. 1.70). In the process, the centre of the sphere moves with velocity v along a circle of radius R . Find the kinetic energy of the sphere.

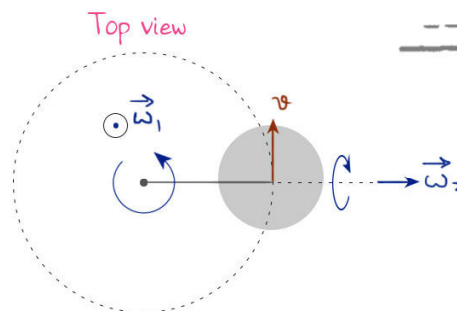


$$KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} \cdot \frac{2}{5} m r^2 (\omega_1^2 + \omega_2^2)$$

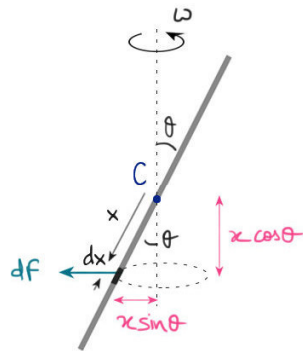
$$= \frac{1}{2} m v^2 + \frac{m r^2}{5} \left[\left(\frac{v}{R} \right)^2 + \left(\frac{v}{r} \right)^2 \right]$$

$$= \frac{m v^2}{10} \left(7 + \frac{2 r^2}{R^2} \right) //$$



Youtube / Arihant Senior

1.269. A midpoint of a thin uniform rod AB of mass m and length l is rigidly fixed to a rotation axle OO' as shown in Fig. 1.71. The rod is set into rotation with a constant angular velocity ω . Find the resultant moment of the centrifugal forces of inertia relative to the point C fixed to the rod. (i.e wrt to rotating frame of reference C)



Centrifugal force w.r.t rotating C point on rod element: $df = dm \omega^2 x \sin \theta$

$$= \frac{dx}{l} m \omega^2 x \sin \theta$$

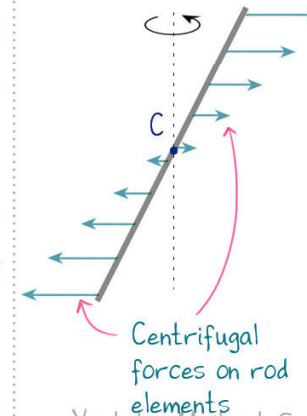
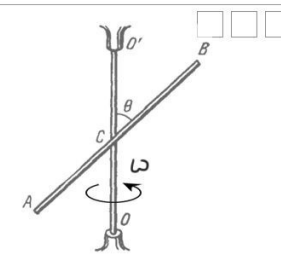
$$= \frac{m \omega^2 \sin \theta}{l} x dx$$

$\therefore$ Torque w.r.t C : $d\tau_c = df x \cos \theta$

$$\tau_c = \frac{m \omega^2 \sin \theta \cos \theta}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{1}{24} m \omega^2 l^2 \sin 2\theta //$$

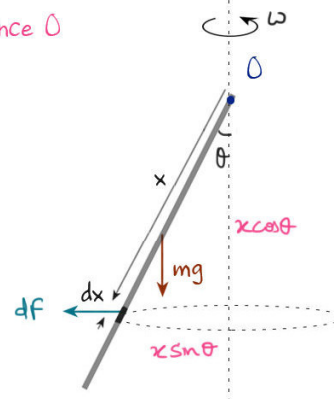
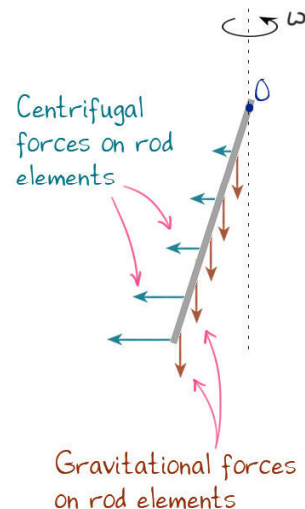
Note: Due to this centrifugal torque, rod tends to be horizontal. But rigid fixed joint at C provides counter torque which prevents that.



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1.270. A conical pendulum, a thin uniform rod of length l and mass m , rotates uniformly about a vertical axis with angular velocity ω (the upper end of the rod is hinged). Find the angle θ between the rod and the vertical.

w.r.t rotating frame of reference O



Torques due to centrifugal force and gravity are balanced. As rod is at rest wrt O .

$$df = dm \omega^2 x \sin \theta$$

$$= \frac{dx}{l} m \omega^2 x \sin \theta$$

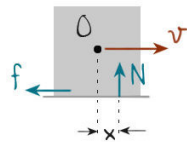
$$mg \left(\frac{l}{2} \sin \theta \right) = \int_0^l df (x \cos \theta)$$

$$\Rightarrow \cancel{mg} \frac{l}{2} \sin \theta = \int_0^l \frac{dx}{l} m \omega^2 x \sin \theta \cos \theta$$

$$\text{Solving, } \cos \theta = \frac{3g}{2l\omega^2} //$$

Youtube / Arihant Senior

1.271. A uniform cube with edge a rests on a horizontal plane whose friction coefficient equals k . The cube is set in motion with an initial velocity, travels some distance over the plane and comes to a standstill. Explain the disappearance of the angular momentum of the cube relative to the axis lying in the plane at right angles to the cube's motion direction. Find the distance between the resultants of gravitational forces and the reaction forces exerted by the supporting plane.



Method 1 w.r.t cm

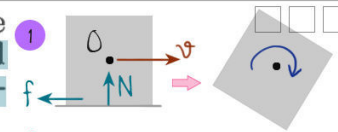
W.r.t center of mass O, cube does not rotate:

$$\therefore \tau_O = 0$$

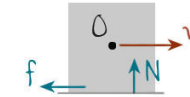
$$\Rightarrow N x = f \frac{a}{2}$$

mg kmg

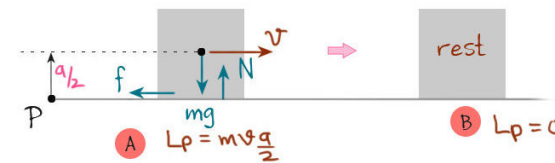
$$\Rightarrow \cancel{mg} x = k \cancel{mg} \frac{a}{2} \Rightarrow x = \frac{ka}{2} //$$



If N passes through O as it normally does, f should rotate the cube (wrt O)



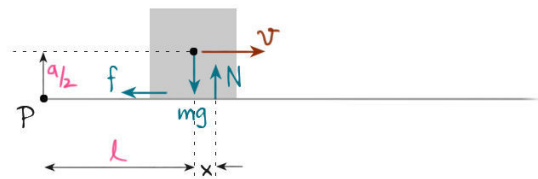
2 $\therefore N$ shifts, provides counter torque and prevents that rotation.



3 $\therefore L_p$ 'disappears', as cube comes to rest at B. As torque due to N neutralises it.

Youtube / Arihant Senior

Method 2 w.r.t general point on plane.



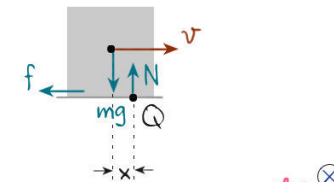
$$\otimes \tau_P = \frac{dL}{dt} \quad \otimes L = \frac{mva}{2}$$

$$mg l - N(l+x) = \frac{ma}{2} \frac{dv}{dt}$$

mg $-kg$

$$\Rightarrow \cancel{mg} x = \cancel{mg} \frac{ka}{2} \Rightarrow x = \frac{ka}{2} //$$

Method 3 w.r.t point of application of N.



$$\otimes \tau_Q = \frac{dL}{dt} \quad \otimes L = \frac{mva}{2}$$

$$-mg x = \frac{ma}{2} \frac{dv}{dt}$$

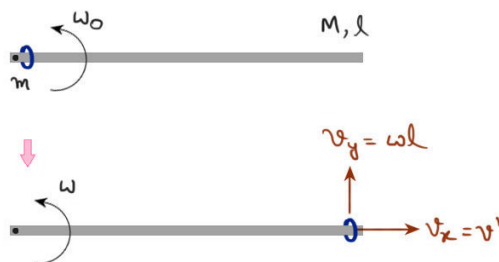
$-kg$

$$\Rightarrow \cancel{mg} x = \cancel{mg} \frac{ka}{2}$$

$$\Rightarrow x = \frac{ka}{2} //$$

Youtube / Arihant Senior

1.272. A smooth uniform rod AB of mass M and length l rotates freely with an angular velocity ω_0 in a horizontal plane about a stationary vertical axis passing through its end A . A small sleeve of mass m starts sliding along the rod from the point A . Find the velocity v' of the sleeve relative to the rod at the moment it reaches its other end B .



KE of m can also be written as:

$$KE_m = KE_R + KE_t$$

$$\frac{1}{2} M l^2 \omega^2 + \frac{1}{2} m v'^2$$

$$L_i = L_f$$

$$\Rightarrow \frac{M l^2}{3} \omega_0 = \left(\frac{M l^2}{3} + m l^2 \right) \omega \quad (1)$$

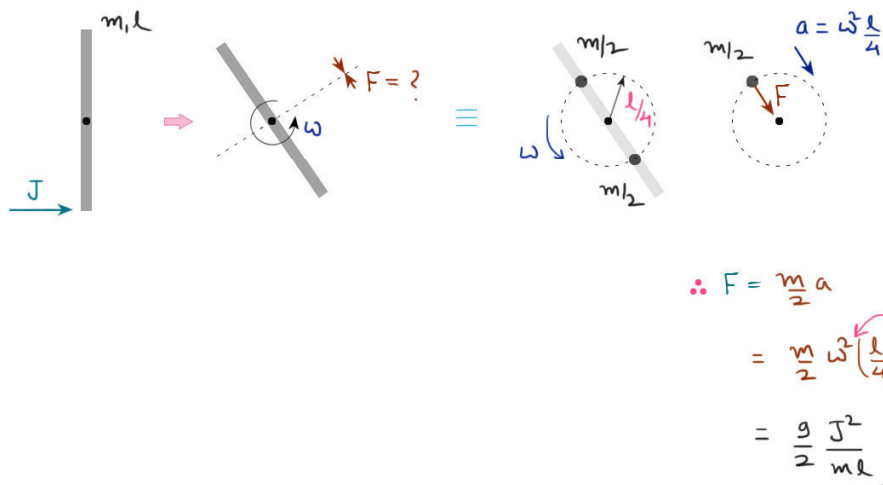
$$KE_i = KE_f$$

$$\frac{1}{2} \frac{M l^2}{3} \omega_0^2 = \underbrace{\frac{1}{2} \frac{M l^2}{3} \omega^2}_{KE_M} + \underbrace{\frac{1}{2} m ((\omega l)^2 + v'^2)}_{KE_m} \quad (2)$$

$$\text{From 1,2: } v' = \frac{l \omega_0}{\sqrt{1 + \frac{3m}{M}}}$$

Youtube/Arihant Senior

1.273. A uniform rod of mass $m = 5.0$ kg and length $l = 90$ cm rests on a smooth horizontal surface. One of the ends of the rod is struck with the impulse $J = 3.0$ N·s in a horizontal direction perpendicular to the rod. As a result, the rod obtains the momentum $p = 3.0$ N·s. Find the force with which one half of the rod will act on the other in the process of motion.



Redundant information

$$\vec{r} \times \vec{J}$$

(Angular (Change in impulse) = angular momentum)

$$\therefore \text{w.r.t } O: J \frac{l}{2} = I \omega$$

$$\Rightarrow J \frac{l}{2} = \frac{m l^2}{12} \omega$$

$$\Rightarrow \omega = \frac{6J}{m l}$$

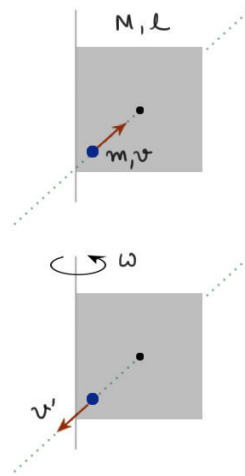
$$\therefore F = \frac{m}{2} a$$

$$= \frac{m}{2} \omega^2 \left(\frac{l}{4} \right)$$

$$= \frac{9}{2} \frac{J^2}{m l}$$

Youtube/Arihant Senior

- 1.274. A thin uniform square plate with side l and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity v at right angles to the plate strikes elastically the centre of it. Find:
- the velocity of the ball v' after the impact;
 - the horizontal component of the resultant force which the axis will exert on the plate after the impact.



• Elastic: $v_{\text{approach}} = v_{\text{separation}}$

$$1 \quad v = \frac{\omega l}{2} + v'$$

About axis: $L_i = L_f$

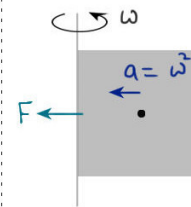
$$2 \quad mv \frac{l}{2} = I\omega - mv' \frac{l}{2}$$

$I = \frac{Ml^2}{3}$

A From 1,2:

$$v' = \frac{4M-3m}{4M+3m}v, \quad \omega = \frac{12mv}{l(3m+4M)}$$

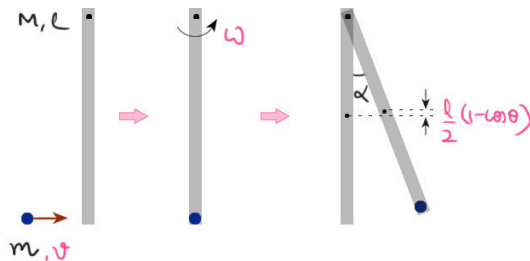
B



$$\begin{aligned} F &= Ma \\ &= M\omega^2 \frac{l}{2} \\ &= \frac{8Mv^2}{l \left(1 + \frac{4M}{3m}\right)^2} \end{aligned}$$

Youtube / Arihant Senior

- 1.275. A vertically oriented uniform rod of mass M and length l can rotate about its upper end. A horizontally flying bullet of mass m strikes the lower end of the rod and gets stuck in it; as a result, the rod swings through an angle α . Assuming that $m \ll M$, find:
- the velocity of the flying bullet;
 - the momentum increment in the system "bullet-rod" during the impact; what causes the change of that momentum;
 - at what distance x from the upper end of the rod the bullet must strike for the momentum of the system "bullet-rod" to remain constant during the impact.



$$L_i = L_f$$

$$mv \approx \frac{Ml^2}{3} \omega \quad 1$$

$$E_i = E_f$$

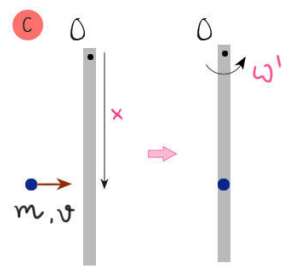
$$\frac{1}{2} \left(\frac{Ml^2}{3} \right) \omega^2 \approx Mg \frac{l(1-\cos\alpha)}{2} \quad 2$$

A From 1,2: $v = \frac{M}{m} \sqrt{\frac{2gl}{3}} \sin \frac{\alpha}{2}, \quad \omega = \sqrt{\frac{6g}{l}} \sin \frac{\alpha}{2}$

B $\Delta p \approx M \omega \frac{l}{2} - mv = M \sqrt{\frac{gl}{6}} \sin \frac{\alpha}{2}$

Momentum change is caused by reaction force on rod by hinge.
Its +ve, therefore hinge force is towards right!

Youtube / Arihant Senior



$$L_i = L_f$$

$$mvx = \frac{Ml^2}{3} \omega'$$

$$\Delta p = 0$$

$$mv = M \omega' \frac{l}{2}$$

$$mv = \frac{Ml(\cancel{mvx})3}{Ml^2} \frac{l}{2}$$

$$\Rightarrow x = \frac{2l}{3} //$$

Depends only on length of rod.
Its independent of m, M, v !!

If bullet hits above $2L/3$: hinge force is towards left.

If bullet hits at $2L/3$: hinge force is zero. (Or point O's initial velocity would be zero even without hinge)

If bullet hits below $2L/3$: hinge force is towards right.

Youtube / Arihant Senior

1.276. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m . A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0 . Then by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find:

- the angular velocity of the system in its final state;
- the work performed by the force F .

We can conserve L about O, as Tension passes through it.

v of ball is zero when it reaches O, as thread is pulled slowly and only external work is due to F ,

$$\text{A} \quad \therefore L_i = L_f$$

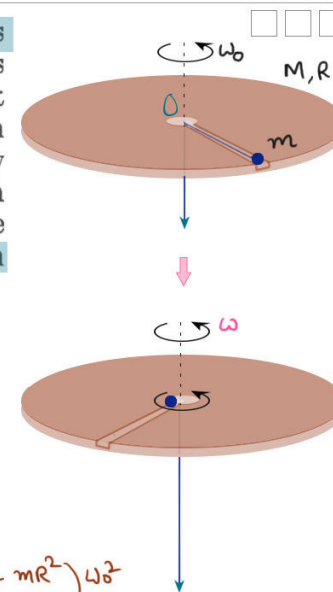
$$\Rightarrow \frac{MR^2}{2} \omega_0 + mR^2 \omega_0 = \frac{MR^2}{2} \omega$$

$$\Rightarrow \omega = \omega_0 \left(1 + \frac{2m}{M}\right) //$$

$$\text{B} \quad \therefore W_F = \Delta KE$$

$$= \frac{1}{2} \left(\frac{MR^2}{2}\right) \omega^2 - \frac{1}{2} \left(\frac{MR^2}{2} + mR^2\right) \omega_0^2$$

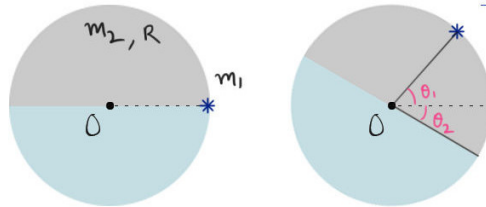
$$= \frac{m\omega_0^2 R^2}{2} \left(1 + \frac{2m}{M}\right) //$$



Youtube / Arihant Senior

1.277. A man of mass m_1 stands on the edge of a horizontal uniform disc of mass m_2 and radius R which is capable of rotating freely about a stationary vertical axis passing through its centre. At a certain moment the man starts moving along the edge of the disc; he shifts over an angle ϕ' relative to the disc and then stops. In the process of motion the velocity of the man varies with time as $v'(t)$. Assuming the dimensions of the man to be negligible, find: w.r.t disc

- (a) the angle through which the disc had turned by the moment the man stopped;
 (b) the force moment (relative to the rotation axis) with which the man acted on the disc in the process of motion.



Sol: A Given: $\theta_1 + \theta_2 = \phi' \quad 1$

About O: $I_1 \theta_1 = I_2 \theta_2$
 $m_1 R^2 \theta_1 = \frac{m_2 R^2}{2} \theta_2 \quad 2$

From 1, 2: $\theta_2 = \frac{2m_1 \phi'}{m_2 + 2m_1} //$

B $\tau_{disc} = I_{disc} \alpha = I \frac{d^2 \theta_2}{dt^2} = I \frac{d}{dt} \left(\frac{d\theta_2}{dt} \right) = \frac{m_2 R^2}{2} \frac{2m_1}{m_2 + 2m_1} \frac{d}{dt} \left(\frac{R d\phi'}{dt} \right) = \frac{m_1 m_2 R}{m_2 + 2m_1} \frac{dv'(t)}{dt} //$

Internal forces:

$$\Delta L = 0$$

$$\Rightarrow I_1 \omega_1 = I_2 \omega_2$$

$$\Rightarrow I_1 \frac{d\theta_1}{dt} = I_2 \frac{d\theta_2}{dt}$$

$$\Rightarrow I_1 \int d\theta_1 = I_2 \int d\theta_2$$

$$\Rightarrow I_1 \theta_1 = I_2 \theta_2$$

Similar to CoM at rest when: $m_A x_A = m_B x_B$

1.278. Two horizontal discs rotate freely about a vertical axis passing through their centres. The moments of inertia of the discs relative to this axis are equal to I_1 and I_2 , and the angular velocities to ω_1 and ω_2 . When the upper disc fell on the lower one, both discs began rotating, after some time, as a single whole (due to friction). Find:

- (a) the steady-state angular rotation velocity of the discs;
 (b) the work performed by the friction forces in this process.

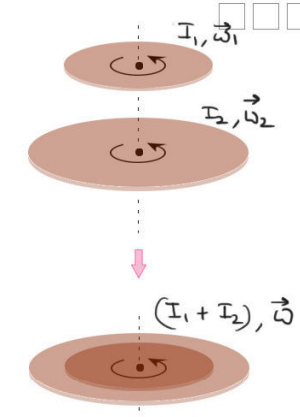
B $W_f = \Delta K E$

$$= \frac{1}{2} (I_1 + I_2) \omega^2 - \frac{1}{2} I_1 \omega_1^2 - \frac{1}{2} I_2 \omega_2^2$$

$$= \frac{1}{2} (I_1 + I_2) \left[\frac{I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + 2 I_1 I_2 \vec{\omega}_1 \cdot \vec{\omega}_2}{(I_1 + I_2)^2} \right] - \frac{1}{2} I_1 \omega_1^2 - \frac{1}{2} I_2 \omega_2^2$$

$$= \frac{I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + 2 I_1 I_2 \vec{\omega}_1 \cdot \vec{\omega}_2 - I_1^2 \omega_1^2 - I_1 I_2 \omega_1^2 - I_2^2 \omega_2^2 - I_1 I_2 \omega_2^2}{2(I_1 + I_2)}$$

$$= \frac{-I_1 I_2}{2(I_1 + I_2)} (\omega_1^2 + \omega_2^2 - 2 \vec{\omega}_1 \cdot \vec{\omega}_2) = \frac{-I_1 I_2}{2(I_1 + I_2)} |\vec{\omega}_1 - \vec{\omega}_2|^2 //$$



A $L_i = L_f$

$$I_1 \vec{\omega}_1 + I_2 \vec{\omega}_2 = (I_1 + I_2) \vec{\omega}$$

$$\Rightarrow \vec{\omega} = \frac{I_1 \vec{\omega}_1 + I_2 \vec{\omega}_2}{I_1 + I_2} //$$

Youtube/Arihant Senior

1.279. A small disc and a thin uniform rod of length l , whose mass is η times greater than the mass of the disc, lie on a smooth horizontal plane. The disc is set in motion, in horizontal direction and perpendicular to the rod, with velocity v , after which it elastically collides with the end of the rod. Find the velocity of the disc and the angular velocity of the rod after the collision. At what value of η will the velocity of the disc after the collision be equal to zero? reverse its direction?

• Elastic: $v_{\text{approach}} = v_{\text{separation}}$

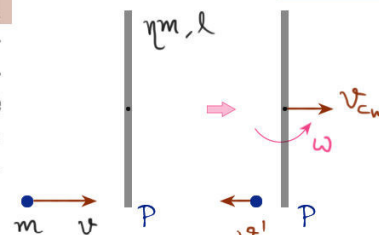
$$v = (v_{\text{cm}} + \frac{\omega l}{2}) + v' \quad (1)$$

About P: $L_i = L_f$

$$0 = I_{\text{cm}} \omega + M \vec{r} \times \vec{v}_{\text{cm}}$$

$$\Rightarrow M \frac{l^2}{12} \omega = M v_{\text{cm}} \frac{l}{2}$$

$$\Rightarrow \omega l = 6 v_{\text{cm}} \quad (2)$$



$$P_i = P_f$$

$$mv = \eta m v_{\text{cm}} - m v'$$

$$v = \eta v_{\text{cm}} - v' \quad (3)$$

We know,

$$L_0 = \vec{r}_{\text{cm}} \times M \vec{v}_{\text{cm}} = I_{\text{cm}} \vec{\omega}$$

From 1,2,3:

$$v' = \frac{\eta - 4}{\eta + 4} v$$

$$\omega = \frac{12v}{l(\eta + 4)}$$

at $\eta = 4$, $v' = 0$

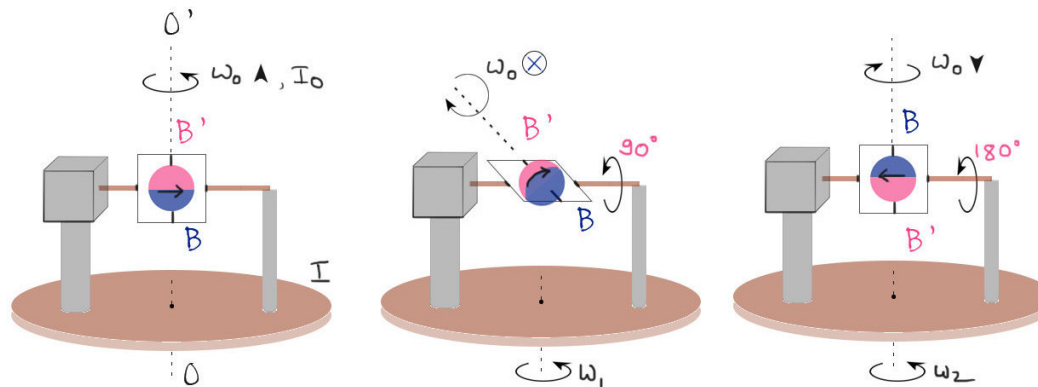
$\eta > 4$, rebound

$\eta < 4$, same direction
Youtube/Arihant Senior

1.280 A sphere (moment of inertia I_0) rotates with ω_0 about BB' inside a square frame. Frame can be rotated by a motor. Motor itself is mounted on a stationary platform that can freely rotate about OO' . Moment of inertia of motor platform system is I about OO' . Find

(a) the work performed by the motor in turning the shaft BB' through 90° ; through 180° ;

(b) the moment of external forces which maintains the axis of the arrangement in the vertical position after the motor turns the shaft BB' through 90° .



A About OO' : $L_i = L_f$

$$90^\circ: I_0 \omega_0 = (I + I_0) \omega_1$$

$$W = \Delta KE$$

$$= \frac{1}{2} (I + I_0) \omega_1^2$$

$$= \frac{1}{2} \frac{I_0^2 \omega_0^2}{(I + I_0)}$$

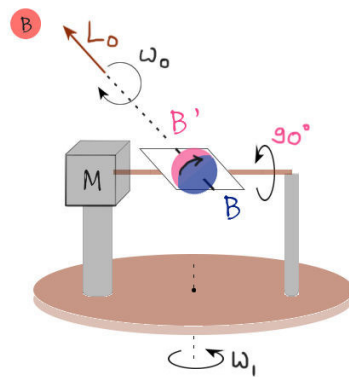
180°: $L_i = L_f$

$$I_0 \omega_0 = I \omega_2 - I_0 \omega_0$$

$$W = \Delta KE = \frac{1}{2} I \omega_2^2$$

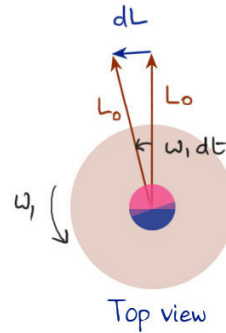
$$= \frac{2 I_0^2 \omega_0^2}{I}$$

Youtube/Arihant Senior



External torque on system is: $\vec{\tau} = \frac{d\vec{L}}{dt}$ 1

Only change in L of system is due to sphere turning along the platform. Its magnitude is constant but direction changes continually.



$$dL = L_0 \omega_1 dt$$

$$\Rightarrow \frac{dL}{dt} = L_0 \omega_1 \quad 2$$

From 1,2: $\vec{\tau} = L_0 \omega_1$

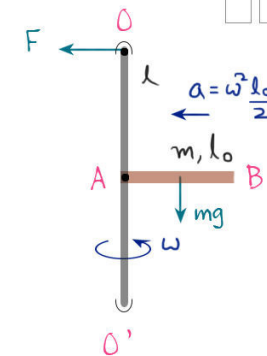
$$= I_0 \omega_0 \frac{I_0 \omega_0}{I + I_0}$$

$$= \frac{I_0^2 \omega_0^2}{I + I_0}$$

Youtube / Arihant Senior

Note: ω_0 never changes in magnitude as w.r.t BB' torque cant be exerted on sphere.

1.281. A horizontally oriented uniform rod AB of mass $m = 1.40$ kg and length $l_0 = 100$ cm rotates freely about a stationary vertical axis OO' passing through its end A. The point A is located at the middle of the axis OO' whose length is equal to $l = 55$ cm. At what angular velocity of the rod the horizontal component of the force acting on the lower end of the axis OO' is equal to zero? What is in this case the horizontal component of the force acting on the upper end of the axis?



Direction of external horizontal component

F must be in direction of a

∴ L is constant, net torque about A is zero:

$$mg \frac{l_0}{2} = F \frac{l}{2}$$

$$\Rightarrow F = mg \frac{l_0}{l}$$

$$F = m \omega^2 \frac{l_0}{2}$$

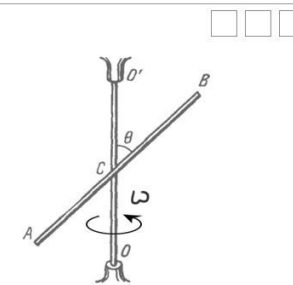
$$\Rightarrow \omega = \sqrt{\frac{2g}{l}}$$

Note: We didnt show vertical components of forces at O and O' as their moment about A will be zero.

- Same with centrifugal force.
- Same with reaction forces at A.

Youtube / Arihant Senior

1.282. The middle of a uniform rod of mass m and length l is rigidly fixed to a vertical axis OO' so that the angle between the rod and the axis is equal to θ (see Fig. 1.71). The ends of the axis OO' are provided with bearings. The system rotates without friction with an angular velocity ω . Find:



- (a) the magnitude and direction of the rod's angular momentum L relative to the point C , as well as its angular momentum relative to the rotation axis;
 (b) how much the modulus of the vector L relative to the point C increases during a half-turn;
 (c) the moment of external forces N acting on the axle OO' in the process of rotation.

A

$$\vec{L}_C = \vec{L}_{\text{axial}} + \vec{L}_{\text{transverse}}$$

$$= I_a \omega \cos \theta + I_t \omega \sin \theta$$

$$\Rightarrow \vec{L}_C = \frac{ml^2}{12} \omega \sin \theta$$

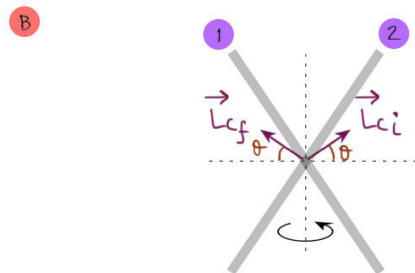
At a given instant, we have broken ω and therefore L_C into two perpendicular components.

M1 L about any axis passing through C = component of L_C along that axis. $\therefore$

$$L_{OO'} = L_C \sin \theta = \frac{ml^2}{12} \omega \sin^2 \theta$$

M2

$$L_{OO'} = I \omega = \frac{m(l \sin \theta)^2}{12} \omega$$



L_C 's magnitude is constant but direction changes continually from 1 to 2.

$\therefore$ In a half turn: $\Delta L = \vec{L}_{cf} - \vec{L}_{ci}$

$$= 2L_C \cos \theta$$

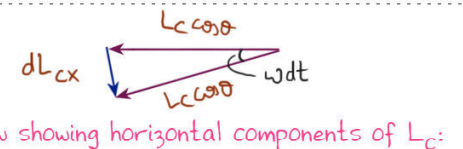
$$= 2 \left(\frac{ml^2}{12} \omega \sin \theta \right) \cos \theta$$

$$= \frac{ml^2}{12} \omega \sin 2\theta$$

- C Breaking L_C in X and Y directions, External torque on system is:

$\therefore$ constant

$$\vec{Z} = \frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{L}_{Cx} + \vec{L}_{Cy})$$



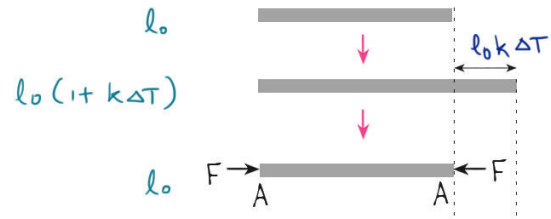
$\therefore dL_{Cx} = L_C \cos \theta \omega dt$

$$\Rightarrow Z = \frac{dL_{Cx}}{dt} = L_C \omega \cos \theta$$

$$= \left(\frac{ml^2}{12} \omega \sin \theta \right) \omega \cos \theta$$

$$= \frac{ml^2}{24} \omega^2 \sin 2\theta$$

1.290. What pressure has to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100°C ?



$$\gamma = \frac{F/A}{\Delta l/l} = \frac{P}{\Delta l/l} = \frac{P}{\frac{l_0 k \Delta T}{l_0(1 + k \Delta T)}}$$

$$\Rightarrow P = \frac{\gamma k \Delta T}{1 + k \Delta T} = \gamma k \Delta T //$$

Youtube / Arihant Senior

1.291. What internal pressure (in the absence of an external pressure) can be sustained

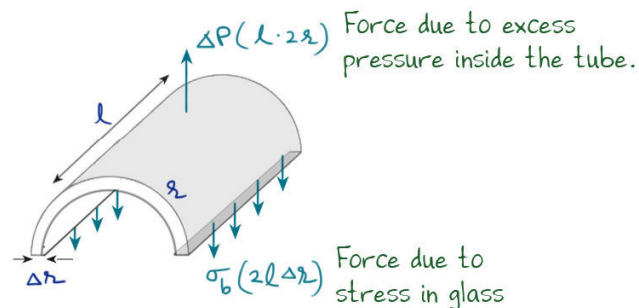
(a) by a glass tube; (b) by a glass spherical flask, if in both cases the wall thickness is equal to $\Delta r = 1.0 \text{ mm}$ and the radius of the tube and the flask equals $r = 25 \text{ mm}$? Breaking stress of glass σ_b is given.

$$F = \Delta P \times A$$

due to gas pressure

$$F = \sigma \times A$$

A Lets consider 1 length of half cylinder.

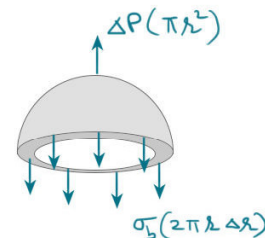


• Its at rest. Forces are equal.

$$\Delta P(l \cdot 2r) = \sigma_b 2l\Delta r$$

$$\Rightarrow \Delta P = \frac{\sigma_b \Delta r}{r} //$$

B Lets consider hemisphere.



$$\Delta P \pi r^2 = \sigma_b (2\pi r \Delta r)$$

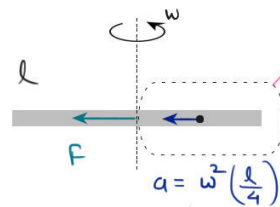
$$\Rightarrow \Delta P = \frac{2 \sigma_b \Delta r}{r} //$$

Clearly, a spherical vessel can sustain double the pressure than a cylindrical vessel.

Youtube / Arihant Senior

1.292. A horizontally oriented copper rod of length $l = 1.0$ m is rotated about a vertical axis passing through its middle. What is the number of rps at which this rod ruptures? σ_b of Cu is given.

If rod ruptures, it will be at the middle.
As there the centripetal force is maximum.



centripetal acceleration of half rod.

On this system:

$$F = \left(\frac{m}{2}\right) a$$

$$\Rightarrow F = \left(\frac{\rho l A}{2}\right) \omega^2 \frac{l}{4}$$

$$\Rightarrow \frac{F}{A} = \frac{\rho l^2 \omega^2}{8}$$

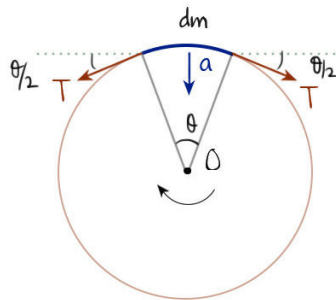
At breaking point $\omega = \omega_{\max}$ and, $F/A = \sigma_b$

$$\therefore \omega_{\max} = \frac{2\sqrt{2}}{l} \sqrt{\frac{\sigma_b}{\rho}}$$

$$\text{In rps: } \omega_{\max} = \frac{1}{2\pi} \left(\frac{2\sqrt{2}}{l} \sqrt{\frac{\sigma_b}{\rho}} \right) \text{ rev/s}$$

Youtube / Arihant Senior

1.293. A ring of radius $r = 25$ cm made of lead wire is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring. What is the number of rps at which the ring ruptures? σ_b is given.



$$F = ma$$

$$\Rightarrow 2T \sin \frac{\theta}{2} = dm \omega^2 r$$

$$\downarrow \rho dV$$

$$\Rightarrow T \theta = (\rho \cdot (\cancel{2} \theta) A) \omega^2 r$$

$$\Rightarrow \frac{T}{A} = \rho r^2 \omega^2$$

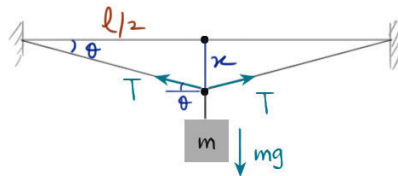
At breaking point $\omega = \omega_{\max}$ and, $T/A = \sigma_b$

$$\therefore \omega_{\max} = \frac{1}{r} \sqrt{\frac{\sigma_b}{\rho}}$$

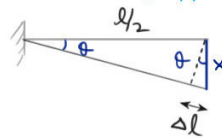
$$\text{In rps: } \omega_{\max} = \frac{1}{2\pi} \left(\frac{1}{r} \sqrt{\frac{\sigma_b}{\rho}} \right) \text{ rev/s}$$

Youtube / Arihant Senior

1.294. A steel wire of diameter $d = 1.0$ mm is stretched horizontally between two clamps located at the distance $l = 2.0$ m from each other. A weight of mass $m = 0.25$ kg is suspended from the midpoint O of the wire. What will the resulting descent of the point O be in centimetres? Y of steel is given.



Note: Wrong approach to find Δl



$$\Delta l \approx x \theta = \frac{l \theta^2}{2} = \frac{2x^2}{l} \quad \times$$

Wrong since its approximation on approximation.

When m comes to rest:

$$2T \sin \theta = mg$$

$$\Rightarrow T = \frac{mg}{2 \sin \theta} = \frac{YA \Delta l}{l/2}$$

$$\sin \theta \rightarrow \frac{x}{l/2}, A \rightarrow \frac{\pi d^2}{4}, \Delta l \rightarrow \frac{x^2}{l}$$

$$\Rightarrow \frac{mg}{2 \cdot \frac{x}{l/2}} = \frac{Y \pi d^2}{4} \cdot \frac{2x^2}{l^2}$$

$$\Rightarrow x = \left(\frac{mgl^3}{2\pi Y d^2} \right)^{1/3}$$

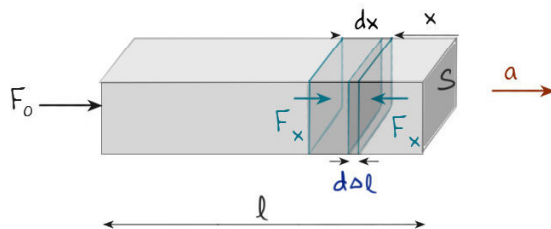
Extension in half wire is given by:

$$\begin{aligned} \Delta l &= \sqrt{\left(\frac{l}{2}\right)^2 + x^2} - \frac{l}{2} \\ &= \frac{l}{2} \left(1 + \left(\frac{2x}{l}\right)^2\right)^{1/2} - \frac{l}{2} \\ &\approx \frac{l}{2} \left(1 + \frac{1}{2} \frac{4x^2}{l^2}\right) - \frac{l}{2} \\ &= \frac{l}{2} + \frac{x^2}{l} - \frac{l}{2} \\ &= \frac{x^2}{l} \end{aligned}$$

Youtube / Arihant Senior

1.295. A uniform elastic plank moves over a smooth horizontal plane due to a constant force F_0 distributed uniformly over the end face. The surface of the end face is equal to S , and Young's modulus of the material is Y . Find the compressive strain of the plank in the direction of the acting force.

Let F_x be the Force at distance x from the end
And the compression in dx be $d\Delta l$



$\therefore$ On section dx :

$$\frac{F_x}{S} = Y \frac{d\Delta l}{dx}$$

$$\Rightarrow \frac{F_0}{lS} \int_0^l x dx = Y \int_0^l d\Delta l$$

$$\Rightarrow \frac{\Delta l}{l} = \frac{F_0}{2YS}$$

$$F = ma$$

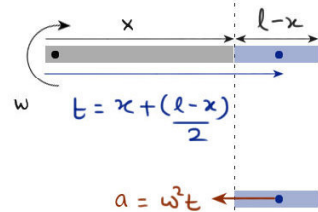
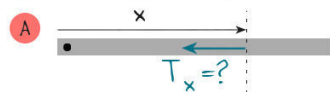
$$F_x = \frac{x}{l} m a$$

$$= \frac{x}{l} m \left(\frac{F_0}{m} \right)$$

$$= \frac{F_0 x}{l}$$

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1.296. A thin uniform copper rod of length l and mass m rotates uniformly with an angular velocity ω in a horizontal plane about a vertical axis passing through one of its ends. Determine the tension in the rod as a function of the distance r from the rotation axis. Find the elongation of the rod. ρ, γ are known.



$\therefore F = ma$ on this section of rod:

$$\Rightarrow T_x = \left(\frac{l-x}{l}\right)m \cdot \omega^2 t = \frac{(l^2 - x^2)}{2l} m \omega^2 //$$

Let the extension in dx be $d\Delta l$

B On section dx :

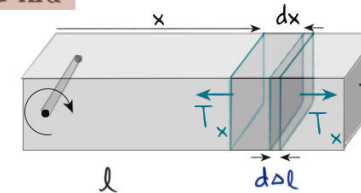
$$\frac{T_x}{S} = \gamma \frac{d\Delta l}{dx} \Rightarrow \frac{T_x}{S} = \gamma \frac{d\Delta l}{dx}$$

$$m = \rho S l$$

$$\Rightarrow \frac{\rho \omega^2}{2l} \int_0^l (l^2 - x^2) dx = \gamma \int_0^l d\Delta l$$

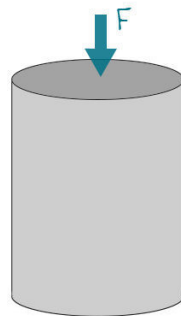
$$\Rightarrow \frac{\rho \omega^2}{2\gamma} \left(l^3 - \frac{l^3}{3} \right) = \Delta l$$

$$\Rightarrow \Delta l = \frac{\rho \omega^2 l^3}{3\gamma} //$$



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1.297. A solid copper cylinder of length $l = 65$ cm is placed on a horizontal surface and subjected to a vertical compressive force $F = 1000$ N directed downward and distributed uniformly over the end face. What will be the resulting change of the volume of the cylinder in cubic millimetres? μ, γ are known.



$$V = \left(\pi \frac{d^2}{4}\right) l$$

$$\Rightarrow \frac{\Delta V}{V} = 2 \frac{\Delta d}{d} + \frac{\Delta l}{l} = 2 \left(-\mu \frac{\Delta l}{l}\right) + \frac{\Delta l}{l}$$

$$\Rightarrow \frac{\Delta V}{V} = \frac{\Delta l}{l} (1 - 2\mu)$$

$$= \frac{F}{A\gamma} (2\mu - 1)$$

$$\Rightarrow \Delta V = \frac{F l}{\gamma} (2\mu - 1) //$$

Poissons ratio:

$$\mu = -\frac{\Delta d/d}{\Delta l/l}$$

In compression:

$$\gamma = \frac{F/A}{-\Delta l/l}$$

Note: Effect of mg would have already happened when its placed on floor, and it wont effect further change in volume when force is applied.

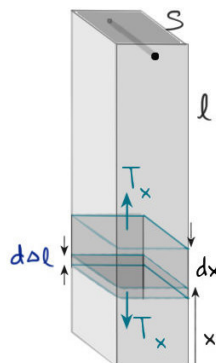
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1.298. A copper rod of length l is suspended from the ceiling by one of its ends. Find:

- (a) the elongation Δl of the rod due to its own weight;
 (b) the relative increment of its volume $\Delta V/V$. β, γ are known.

A Let T_x be the tension at distance x from the end

And extension in dx be $d\Delta l$



∴ On section dx :

$$\frac{T_x}{S} = \gamma \frac{d\Delta l}{dx}$$

$$\Rightarrow \frac{mg}{lS} \int_0^l x dx = \gamma \int_0^{\Delta l} d\Delta l$$

$$\Rightarrow \Delta l = \frac{mg}{lS\gamma} \frac{l^2}{2} = \frac{\beta \gamma l^2}{2\gamma} //$$

$$T_x = \left(\frac{x}{l}m\right)g$$

(Proved in 1.297)

$$B \quad \frac{\Delta V}{V} = \frac{\Delta l}{l} (1 - 2\mu) //$$

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1.299. A bar made of material whose Young's modulus is equal to γ and Poisson's ratio to μ is subjected to the hydrostatic pressure p . Find:

- (a) the fractional decrement of its volume;
 (b) the relationship between the compressibility β and the elastic constants γ and μ .

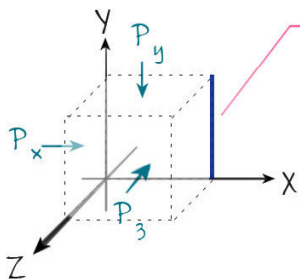
Show that Poisson's ratio μ cannot exceed $1/2$.

Poisson's ratio:

$$\mu = \frac{-\Delta d/d}{\Delta l/l}$$

$$\gamma = \frac{P}{-\Delta l/l}$$

A Lets consider a cubic portion from the bar.



Change in this side of the cube will be:

- ① Δl , due to P_y (will reduce)
- ② Δd_1 , due to P_x (will increase)
- ③ Δd_2 , due to P_z (will increase)

All three effects will be independent.

$$V = l^3$$

$$\Rightarrow \frac{\Delta V}{V} = \frac{3\Delta l}{l} \text{ ①}$$

$$\therefore \Delta l = \Delta l_1 + \Delta d_1 + \Delta d_2$$

$$\Rightarrow \frac{\Delta l}{l} = \frac{\Delta l_1}{l} + \frac{\Delta d_1}{d} + \frac{\Delta d_2}{d}$$

∴ its cube, $l = d$

$$\Rightarrow \frac{\Delta l}{l} = \frac{-p}{\gamma} + \mu \frac{p}{\gamma} + \mu \frac{p}{\gamma} \text{ ②}$$

From 1,2:

$$\frac{\Delta V}{V} = \frac{-3p(1-2\mu)}{\gamma}$$

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B Relationship between: β, γ, μ ?

$$\frac{\Delta V}{V} = -\frac{3P(1-2\mu)}{\gamma} \quad (3)$$

$$\Rightarrow \frac{P}{B} = \frac{3P(1-2\mu)}{\gamma}$$

$$\Rightarrow \beta = \frac{3(1-2\mu)}{\gamma} //$$

Bulk Modulus $B = \frac{P}{-\Delta V/V}$

Compressibility: $\beta = \frac{1}{B}$

C ∴ On increasing pressure, ΔV must be negative
From 3:

$$-(1-2\mu) < 0$$

$$\Rightarrow 2\mu < 1$$

$$\mu < 1/2 // \text{ hence proved}$$

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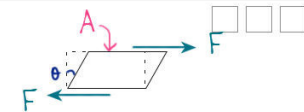
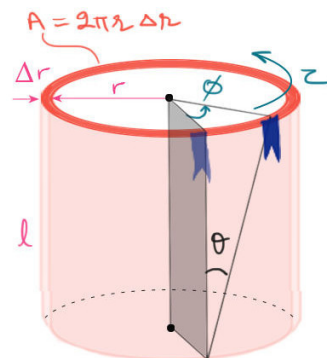
1.305. Determine the relationship between the torque N and the torsion angle ϕ for

(a) the tube whose wall thickness Δr is considerably less than the tube radius;

(b) for the solid rod of circular cross-section. Their length l , radius r , and shear modulus G are supposed to be known.

Lets say the tangential force acting on top surface causes torque τ
It twists the flag on top by ϕ w.r.t center and θ w.r.t bottom.

We need to find relation between τ and ϕ



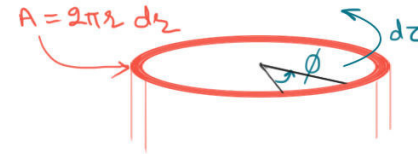
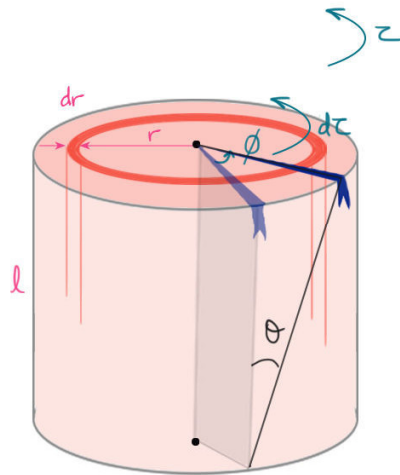
Shear modulus:

$$G = \frac{F/A}{\theta}$$

$$\begin{aligned} \tau &= F \cdot r \\ &= \frac{F}{A} \cdot A \cdot r \\ &= G \cdot \theta \cdot A \cdot r \\ &= G \cdot \phi \cdot \frac{2\pi r \Delta r}{l} \cdot r \\ &= G \left(\frac{2\pi r^3 \Delta r}{l} \right) \phi \end{aligned}$$

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- Let the total torque be τ
 Then, on cylindrical element, its $d\tau$
 Also, all cylindrical elements are twisted by same angle ϕ



∴ For a cylindrical shell: $d\tau = \frac{G(2\pi r^3 dr)\phi}{l}$

$$\Rightarrow \tau = \frac{2\pi G \phi}{l} \int_0^R r^3 dr$$

$$= \frac{\pi G R^4}{2l} \phi$$

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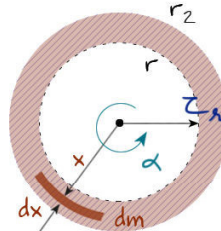
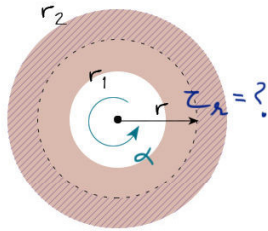
- 1.306. Calculate the torque N twisting a steel tube of length $l = 3.0$ m through an angle $\phi = 2.0^\circ$ about its axis, if the inside and outside diameters of the tube are equal to $d_1 = 30$ mm and $d_2 = 50$ mm.

$$\tau = \frac{2\pi G \phi}{l} \int_{d_1/2}^{d_2/2} r^3 dr = \frac{\pi G \phi (d_2^4 - d_1^4)}{32l}$$

- 1.307. Find the maximum power which can be transmitted by means of a steel shaft rotating about its axis with an angular velocity $\omega = 120$ rad/s, if its length $l = 200$ cm, radius $r = 1.50$ cm, and the permissible torsion angle $\phi = 2.5^\circ$.

$$P = \tau \omega = \left(\frac{2\pi G \phi}{l} \int_0^R r^3 dr \right) \omega = \frac{\pi G \phi R^4}{2l} \omega$$

1.308. A uniform ring of mass m , with the outside radius r_2 , is fitted tightly on a shaft of radius r_1 . The shaft is rotated about its axis with a constant angular acceleration α . Find the moment of elastic forces in the ring as a function of the distance r from the rotation axis.



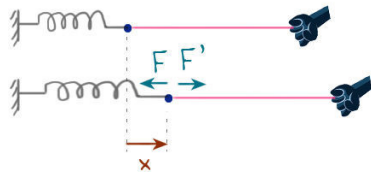
Let the torque at r be τ_r

Now, τ_r is responsible for rotating the outer part of the ring (shaded portion) as other interactions within outer part of ring will be internal.

$$\begin{aligned} \tau_r &= I \alpha \\ &= \int dm r^2 \alpha \\ &= \alpha \int_{r_1}^{r_2} \frac{2\pi r dr}{\pi(r_2^2 - r_1^2)} \cdot M r^2 \\ &= \frac{2M\alpha}{(r_2^2 - r_1^2)} \int_{r_1}^{r_2} r^3 dr \\ &= \frac{2M\alpha}{r_2^2 - r_1^2} \left[\frac{r^4}{4} \right]_{r_1}^{r_2} \end{aligned}$$

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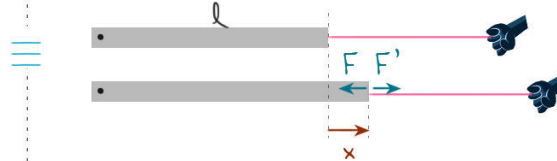
Elastic potential energy per unit volume



Force by spring is: $F = -kx$ ①

• In stretching the spring by dx :

$$\begin{aligned} \Rightarrow dU &= -F \cdot dx \\ \Rightarrow &= -(-kx) dx \\ \Rightarrow U &= k \int_0^x x dx \\ &= \frac{kx^2}{2} \text{ ②} \end{aligned}$$



In a rod: $\frac{F'}{A} = Y \frac{x}{l}$

$$\begin{aligned} \Rightarrow \frac{F}{A} &= Y \frac{x}{l} \\ \Rightarrow F &= -\frac{AY}{l} x \end{aligned}$$

Similar to ①

• Similar to ②

$$\begin{aligned} U &= \frac{AY}{l} \cdot \frac{\Delta l^2}{2} \\ &= \frac{AYl}{2} \left(\frac{\Delta l}{l} \right)^2 \\ \Rightarrow \frac{U}{Al} &= \frac{Y}{2} \epsilon^2 \end{aligned}$$

• Elastic deformation energy per unit volume is:

$$\frac{1}{2} Y \epsilon^2 = \frac{1}{2} \sigma \epsilon = \frac{\sigma^2}{2Y}$$

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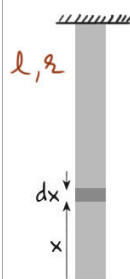
1.309. Find the elastic deformation energy of a steel rod of mass $m = 3.1$ kg stretched to a tensile strain $\epsilon = 1.0 \cdot 10^{-3}$.

$$\frac{U}{V} = \frac{Y\epsilon^2}{2} \Rightarrow U = \left(\frac{Y\epsilon^2}{2}\right) \frac{m}{\rho}$$

1.310. A steel cylindrical rod of length l and radius r is suspended by its end from the ceiling.

- (a) Find the elastic deformation energy U of the rod.
(b) Define U in terms of tensile strain $\Delta l/l$ of the rod.

A dU stored in dx is given by:



$$\begin{aligned} \frac{dU}{dV} &= \frac{\sigma^2}{2Y} \\ \Rightarrow dU &= \frac{1}{2Y} \left(\frac{F_x}{A}\right)^2 dV \\ &= \frac{1}{2Y} \left(\frac{m_x g}{A}\right)^2 A dx \\ &\quad \{m_x = \rho A x\} \end{aligned}$$

$$\begin{aligned} \Rightarrow dU &= \frac{1}{2Y} (\rho x g)^2 \pi r^2 dx \\ \Rightarrow U &= \frac{\pi \rho^2 g^2 r^2}{2Y} \int_0^l x^2 dx \\ &= \frac{\pi \rho^2 g^2 r^2 l^3}{6Y} \end{aligned}$$

B From 1.298, for a hanging rod:

$$\Delta l = \frac{\rho g l^2}{2Y}$$

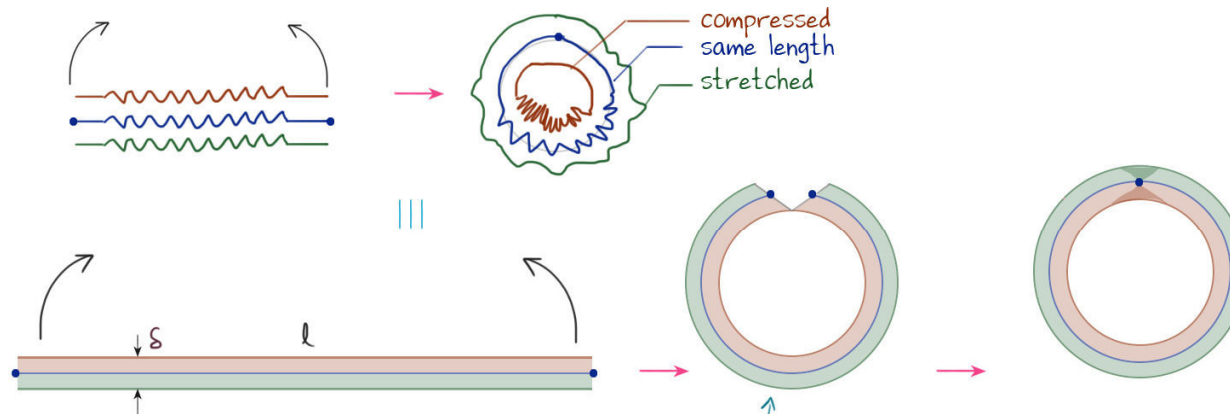
From 1,2:

Writing ρ in terms of $\frac{\Delta l}{l}$

$$U = \frac{2}{3} \pi Y r^2 l \epsilon^2$$

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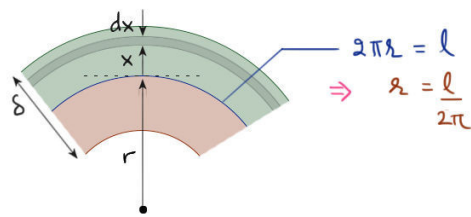
1.311. What work has to be performed to make a hoop out of a steel band of length $l = 2.0$ m, width $h = 6.0$ cm, and thickness $\delta = 2.0$ mm? The process is assumed to proceed within the elasticity range of the material. Hoop is made by bending the 'length'. Width remains same.



At this stage, already
outer portion has expanded a bit
and inner compressed a bit.
• Some work is already done.

When hoop is further completed,
— middle line remains of same length
— outer region will stretch further
— inner region will compress further.

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dU stored in cylindrical shell of dx thickness is given by:

$$\frac{dU}{dV} = \frac{\gamma \epsilon^2}{2}$$

$$\Rightarrow dU = \frac{\gamma}{2} \left(\frac{\Delta l}{l} \right)^2 dV$$

$$= \frac{\gamma}{2} \left(\frac{2\pi(r+x) - 2\pi r}{l} \right)^2 \cdot 2\pi(r+x) dx h$$

$$= \frac{\gamma}{2} \left(\frac{4\pi^2 x^2}{l^2} \right) 2\pi(r+x) dx h$$

$$\Rightarrow U = \frac{4\pi^3 \gamma h}{l^2} \int_{-\delta/2}^{+\delta/2} (rx^2 + x^3) dx$$

$$= \frac{4\pi^3 \gamma h}{l^2} \left[\frac{r}{3} \left(\frac{\delta^3}{4} \right) \right] = \frac{\pi^2 \gamma h \delta^3}{6l}$$

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1.312. Find the elastic deformation energy of a steel rod whose one end is fixed and the other is twisted through an angle $\phi = 6.0^\circ$. The length of the rod is equal to $l = 1.0$ m, and the radius to $r = 10$ mm.

$$\tau = \frac{\pi G R^4}{2l} \phi \quad (\text{Derived in 1.305 B})$$

where, $G \rightarrow$ Shear modulus

$$\therefore U = \frac{C \phi^2}{2} = \frac{\pi G r^4 \phi^2}{4l}$$

$$\text{If } \tau = -c\phi$$

$$dU = -\tau d\phi$$

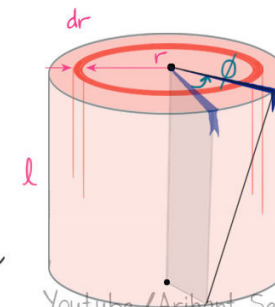
$$\Rightarrow U = c \int \phi d\phi$$

$$\Rightarrow U = \frac{c \phi^2}{2}$$

1.313. Find how the volume density of the elastic deformation energy is distributed in a steel rod depending on the distance r from its axis. The length of the rod is equal to l , the torsion angle to ϕ .

$$\text{Diff. 1} \quad \frac{dU}{dr} = \frac{\pi G \phi^2 r^3}{l} \Rightarrow dU = \frac{\pi G \phi^2 r^3}{l} dr$$

$$\Rightarrow \frac{dU}{dV} = \frac{\pi G \phi^2 r^3}{l \cdot 2\pi r dr l} = \frac{G \phi^2 r^2}{2l^2}$$



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1.314. Find the volume density of the elastic deformation energy in fresh water at the depth of $h = 1000$ m. Bulk modulus is B .

Energy density: $\frac{dU}{dV} = \frac{1}{2} \text{ stress} \times \text{strain}$

$$\Rightarrow \frac{dU}{dV} = \frac{1}{2} P \cdot \frac{P}{B}$$

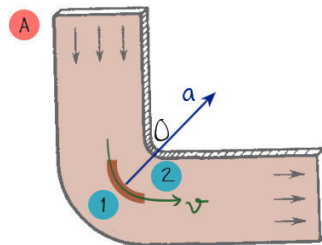
$$= \frac{P^2}{2B}$$

$$= \frac{(\rho gh)^2}{2B} = \frac{(\rho gh)^2}{2} \beta \quad \left(\because B = \frac{1}{\beta} \right) \text{ compressibility}$$

Stress = P
Strain = $\frac{\Delta V}{V}$
Bulk Modulus = $B = \frac{P}{\frac{\Delta V}{V}}$

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1.315. Ideal fluid flows along a flat tube of constant cross-section, located in a horizontal plane and bent as shown in Fig. 1.80 (top view). The flow is steady. Are the pressures and velocities of the fluid equal at points 1 and 2? What is the shape of the streamlines?

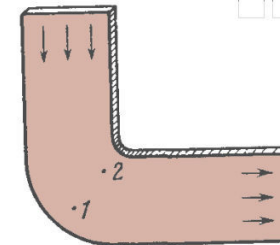


Lets take a thin layer of liquid passing through the bend.
As its turning, its centripetal acceleration a is towards O .

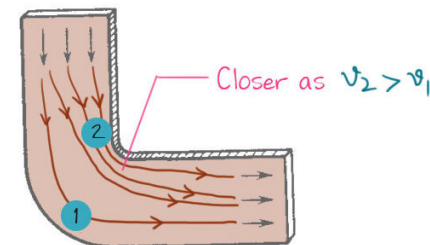
$$\therefore P_1 > P_2$$

$$\Rightarrow P_1 > P_2$$

$$\Rightarrow v_1 < v_2 \quad \left(\because P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2 \right)$$

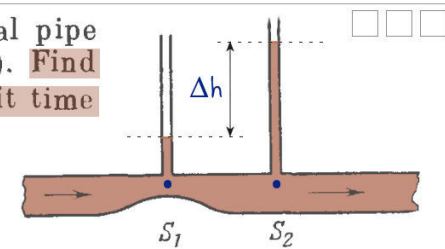


B Streamlines:



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1.316. Two manometric tubes are mounted on a horizontal pipe of varying cross-section at the sections S_1 and S_2 (Fig. 1.81). Find the volume of water flowing across the pipe's section per unit time if the difference in water columns is equal to Δh .



Bernoulli equation:

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2$$

$$\Rightarrow \frac{1}{2} \rho (v_1^2 - v_2^2) = \rho g \Delta h \quad (1) \quad (\because P_2 - P_1 = \rho g \Delta h)$$

Continuity equation:

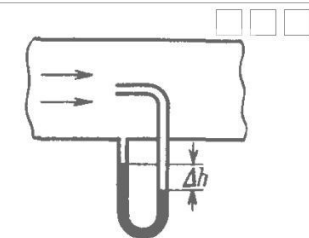
$$v_1 S_1 = v_2 S_2 \quad (2)$$

From 1,2: $v_1 = S_2 \sqrt{\frac{2g \Delta h}{S_2^2 - S_1^2}}$

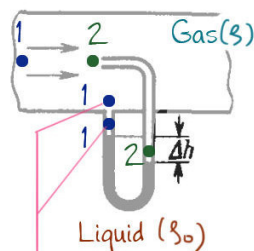
$$\Rightarrow \frac{dV}{dt} = v_1 S_1$$

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1.317. A Pitot tube (Fig. 1.82) is mounted along the axis of a gas pipeline whose cross-sectional area is equal to S . Assuming the viscosity to be negligible, find the volume of gas flowing across the section of the pipe per unit time, if the difference in the liquid columns is equal to Δh , and the densities of the liquid and the gas are ρ_0 and ρ respectively.



Gas flowing in pipe will stop at mouth of pitot, causing rise in pressure there.



$\Delta p = 0$
as we are moving perpendicular to the flow of air.

∴ Pressure change due to gravity in gases is negligible, Pressure at ● is equal and so is pressure at ●.

From pitot tube's liquid:

$$P_2 - P_1 = \rho_0 g \Delta h \quad (1)$$

Bernoulli on gas:

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2 \quad (2)$$

From 1,2:

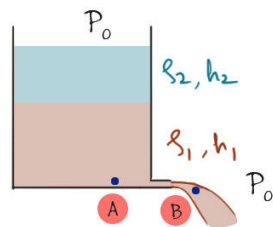
$$\rho_0 g \Delta h = \frac{1}{2} \rho v_1^2$$

$$\Rightarrow v_1 = \sqrt{\frac{2 \rho_0 g \Delta h}{\rho}}$$

$$\therefore \frac{dV}{dt} = S v_1$$

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1.318. A wide vessel with a small hole in the bottom is filled with water and kerosene. Neglecting the viscosity, find the velocity of the water flow, if the thickness of the water layer is equal to $h_1 = 30$ cm and that of the kerosene layer to $h_2 = 20$ cm.



Bernoulli between A and B:

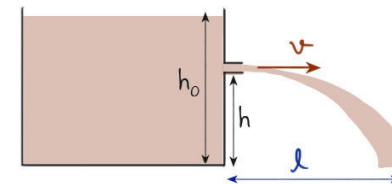
$$P_A + \frac{1}{2} \rho_1 v_A^2 = P_B + \frac{1}{2} \rho_1 v_B^2$$

$$P_0 + \rho_2 g h_2 + \rho_1 g h_1 = P_0 + \frac{1}{2} \rho_1 v_B^2$$

$$\Rightarrow v_B = \sqrt{\frac{2g(\rho_2 h_2 + \rho_1 h_1)}{\rho_1}}$$

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1.319. A wide cylindrical vessel 50 cm in height is filled with water and rests on a table. Assuming the viscosity to be negligible, find at what height from the bottom of the vessel a small hole should be perforated for the water jet coming out of it to hit the surface of the table at the maximum distance l_{max} from the vessel. Find l_{max} .



Velocity of efflux: $v = \sqrt{2g(h_0 - h)}$

Range: $l = v_T \sqrt{\frac{2h}{g}}$
 $= \sqrt{2g(h_0 - h)} \sqrt{\frac{2h}{g}}$

For maximum range: $\frac{dl}{dh} = 0$

$$\Rightarrow \frac{d}{dh} [(h_0 - h)(h)] = 0$$

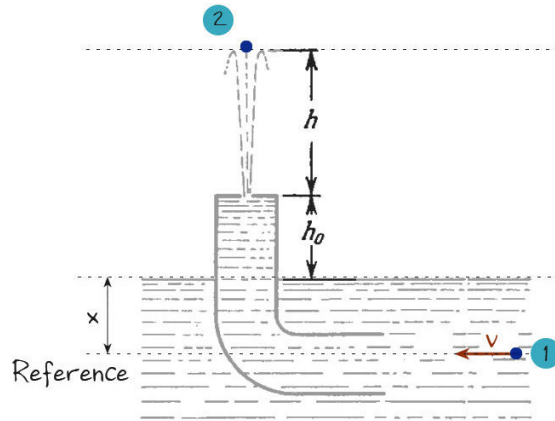
$$\Rightarrow h_0 - 2h = 0$$

$$\Rightarrow h = \frac{h_0}{2}$$

$$\therefore l_{max} = \sqrt{2g(h_0 - \frac{h_0}{2})} \sqrt{\frac{2 \cdot \frac{h_0}{2}}{g}} = h_0$$

Youtube / Arihant Senior

1.320. A bent tube is lowered into a water stream as shown in Fig. 1.83. The velocity of the stream relative to the tube is equal to $v = 2.5$ m/s. The closed upper end of the tube located at the height $h_0 = 12$ cm has a small orifice. To what height h will the water jet spurt?



Bernoulli between 1 and 2:

$$P_1 + \cancel{\rho g h_1} + \frac{1}{2} \cancel{\rho v_1^2} = P_2 + \rho g h_2 + \frac{1}{2} \cancel{\rho v_2^2}$$

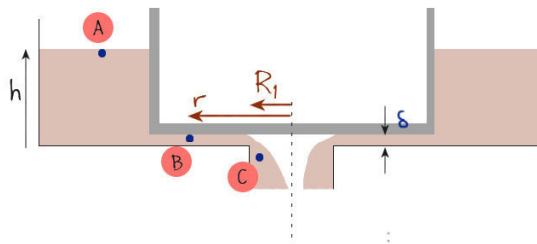
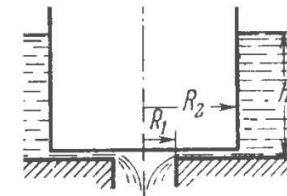
$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 $P_0 + \rho g x$ v P_0 $h + h_0 + x$

$$\Rightarrow \cancel{P_0} + \cancel{\rho g x} + \frac{1}{2} \rho v^2 = \cancel{P_0} + \rho g (h + h_0 + \cancel{x})$$

$$\Rightarrow h = \frac{v^2}{2g} - h_0 //$$

Youtube / Arihant Senior

1.321. The horizontal bottom of a wide vessel with an ideal fluid has a round orifice of radius R_1 over which a round closed cylinder is mounted, whose radius $R_2 > R_1$ (Fig. 1.84). The clearance between the cylinder and the bottom of the vessel is very small, the fluid density is ρ . Find the static pressure of the fluid in the clearance as a function of the distance r from the axis of the orifice (and the cylinder), if the height of the fluid is equal to h .



Bernoulli between A and C:

$$\cancel{P_0} + \cancel{\rho g h} = \cancel{P_0} + \frac{1}{2} \rho v_c^2$$

$$\Rightarrow v_c = \sqrt{2gh}$$

Continuity between B and C:

$$v_B A_B = v_C A_C$$

$$\Rightarrow v_B \cancel{2\pi R_2} = \sqrt{2gh} \cdot \cancel{2\pi R_1}$$

$$\Rightarrow v_B = \sqrt{2gh} \frac{R_1}{R_2}$$

Bernoulli between B and C:

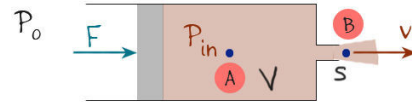
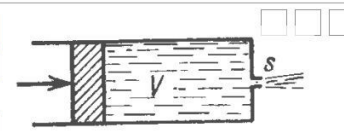
$$P_B + \frac{1}{2} \rho v_B^2 = P_0 + \frac{1}{2} \rho v_c^2$$

$$\Rightarrow P_B = P_0 + \frac{1}{2} \rho (v_c^2 - v_B^2)$$

$$= P_0 + \rho g h \left(1 - \frac{R_1^2}{R_2^2}\right) //$$

Youtube / Arihant Senior

1.322. What work should be done in order to squeeze all water from a horizontally located cylinder (Fig. 1.85) during the time t by means of a constant force acting on the piston? The volume of water in the cylinder is equal to V , the cross-sectional area of the orifice to s , with s being considerably less than the piston area. The friction and viscosity are negligibly small. g is known.



$\therefore v$ is constant

$$\text{Volume flow rate } vs = \frac{V}{t}$$

$$\Rightarrow v = \frac{V}{st}$$

$\therefore$ Piston is assumed massless, forces on it are balanced.

$$F + P_0 A = P_{in} A \Rightarrow P_{in} \text{ is constant}$$

Bernoulli's between A and B:

$$P_{in} = P_0 + \frac{1}{2} \rho v^2 \Rightarrow v \text{ is constant}$$

$$W_{ext} = \Delta KE$$

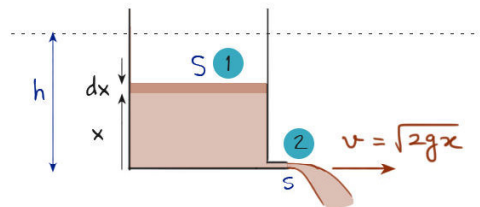
$$= \frac{1}{2} m v^2$$

$$= \frac{1}{2} \rho V \left(\frac{V}{st} \right)^2$$

$$= \frac{1}{2} \frac{\rho V^3}{s^2 t^2}$$

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1.323. A cylindrical vessel of height h and base area S is filled with water. An orifice of area $s \ll S$ is opened in the bottom of the vessel. Neglecting the viscosity of water, determine how soon all the water will pour out of the vessel.



Lets say, water level falls from h to x in time t .

Further at $t + dt$. Continuity at 1 & 2:

$$\frac{dV}{dt} = \frac{dV}{dt}$$

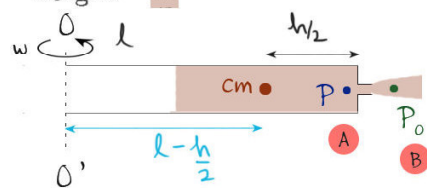
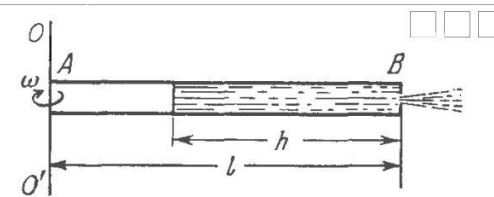
$$\Rightarrow \frac{S(-dx)}{dt} = (\sqrt{2gx}) s$$

$$\Rightarrow \int_0^{t_0} dt = -\frac{S}{s} \frac{1}{\sqrt{2g}} \int_h^0 \frac{dx}{\sqrt{x}}$$

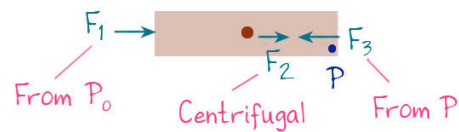
$$\Rightarrow t_0 = \frac{S}{s\sqrt{2g}} 2\sqrt{h}$$

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1.324. A horizontally oriented tube AB of length l rotates with a constant angular velocity ω about a stationary vertical axis OO' passing through the end A (Fig. 1.86). The tube is filled with an ideal fluid. The end A of the tube is open, the closed end B has a very small orifice. Find the velocity of the fluid relative to the tube as a function of the column "height" h .



w.r.t OO' forces on water column:



As orifice is small, water is at rest

$$\therefore F_3 = F_1 + F_2$$

$$\Rightarrow P A = P_0 A + F_2$$

$$\Rightarrow P = P_0 + F_2/A \quad (1)$$

Bernoulli's between A and B:

$$P = P_0 + \frac{1}{2} \rho v^2 \quad (2)$$

Centrifugal force:

$$F_2 = \underbrace{(\rho A h)}_m \omega^2 \left(l - \frac{h}{2} \right) \quad (3)$$

From 1,2:

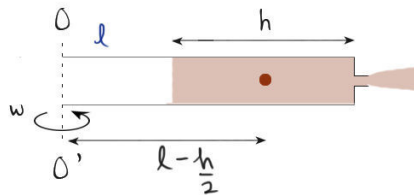
$$P_0 + \frac{F_2}{A} = P_0 + \frac{1}{2} \rho v^2$$

$$\Rightarrow v^2 = \frac{2 F_2}{\rho A}$$

$$\Rightarrow v = \omega \sqrt{2l - h}$$

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Method 2



w.r.t OO' Centrifugal force:

$$F_2 = m \omega^2 \left(l - \frac{h}{2} \right) \equiv m g_{\text{eff}}$$

Velocity of efflux:

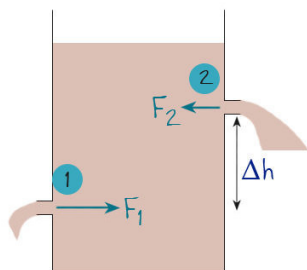
$$v = \sqrt{2 g_{\text{eff}} h}$$

$$= \sqrt{2 \omega^2 \left(l - \frac{h}{2} \right) h}$$

$$= \omega \sqrt{2l - h}$$

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1.326. On the opposite sides of a wide vertical vessel filled with water two identical holes are opened, each having the cross-sectional area $S = 0.50 \text{ cm}^2$. The height difference between them is equal to $\Delta h = 51 \text{ cm}$. Find the resultant force of reaction of the water flowing out of the vessel.



$$\begin{aligned}
 F_{\text{reaction}} &= \frac{dp}{dt} \rightarrow \rho v \\
 &= \frac{d(mv)}{dt} \rightarrow \rho v S \\
 &= \rho v \frac{dV}{dt} \\
 &= \rho v^2 S
 \end{aligned}$$

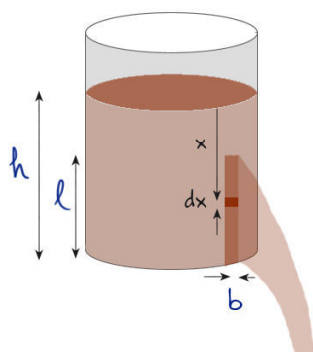
$$\begin{aligned}
 F_{\text{net}} &= F_1 - F_2 \\
 &= \frac{dp_1}{dt} - \frac{dp_2}{dt} \\
 &= \rho S (v_1^2 - v_2^2) \\
 &= \rho S (2gh_1 - 2gh_2) \\
 &= 2\rho g S \Delta h //
 \end{aligned}$$

Another approach for F_{reaction}

$$\begin{aligned}
 F dt &= dp \\
 &= dm v \\
 &= (\rho S dx) v \\
 \Rightarrow F &= \rho S v \frac{dx}{dt} = \rho v^2 S
 \end{aligned}$$

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1.327. The side wall of a wide vertical cylindrical vessel of height $h = 75 \text{ cm}$ has a narrow vertical slit running all the way down to the bottom of the vessel. The length of the slit is $l = 50 \text{ cm}$ and the width $b = 1.0 \text{ mm}$. With the slit closed, the vessel is filled with water. Find the resultant force of reaction of the water flowing out of the vessel immediately after the slit is opened.



$$\begin{aligned}
 F_{\text{reaction}} &= \rho v^2 S \\
 &\text{(Derived in 1.326)}
 \end{aligned}$$

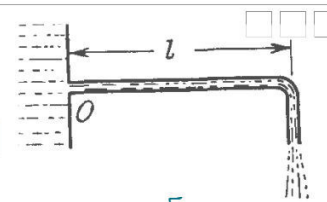
$v \rightarrow$ velocity of efflux
 $S \rightarrow$ area of orifice

• Reaction force from elemental opening of dx height:

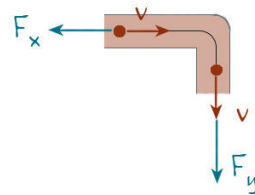
$$\begin{aligned}
 dF &= \rho v^2 (b dx) \\
 &= \rho (2gx) b dx \\
 \Rightarrow F &= 2\rho b g \int_{h-l}^h x dx \\
 &= \rho b g l (2h - l) //
 \end{aligned}$$

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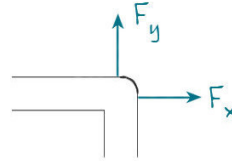
1.328. Water flows out of a big tank along a tube bent at right angles; the inside radius of the tube is equal to $r = 0.50$ cm (Fig. 1.87). The length of the horizontal section of the tube is equal to $l = 22$ cm. The water flow rate is $Q = 0.50$ litres per second. Find the moment of reaction forces of flowing water, acting on the tube's walls, relative to the point O .



Force exerted on water by pipe

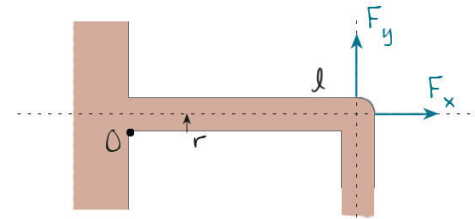


Force exerted on pipe by water



$$F_x = \frac{dp_x}{dt} = v \frac{dm}{dt} = v \frac{d(\rho V)}{dt} = \rho v \frac{dV}{dt} = \rho v Q$$

Similarly: $F_y = \frac{dp_y}{dt} = \rho v Q$



$$\therefore \tau_O = F_y l - F_x r$$

$$= \rho v Q (l - r)$$

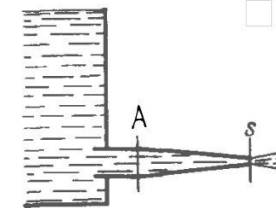
$$= \rho v Q l$$

$Q = \pi r^2 v$

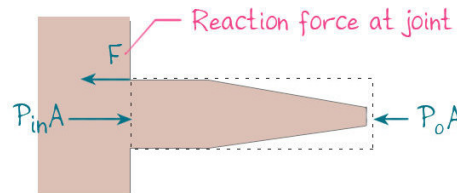
$$= \frac{\rho Q^2 l}{\pi r^2}$$

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1.329. A side wall of a wide open tank is provided with a narrowing tube (Fig. 1.88) through which water flows out. The cross-sectional area of the tube decreases from $A = 3.0$ cm² to $s = 1.0$ cm². The water level in the tank is $h = 4.6$ m higher than that in the tube. Neglecting the viscosity of the water, find the horizontal component of the force tending to pull the tube out of the tank.



If the tube is closed shut forces on pipe-water system are shown

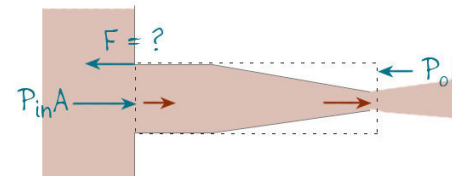


Momentum of system is zero, a constant:

$$\therefore F_{net} = P_{in} A - F - P_o A = 0$$

$$\Rightarrow F = (P_{in} - P_o) A$$

If the tube is open, external forces on pipe-water system are still these three.



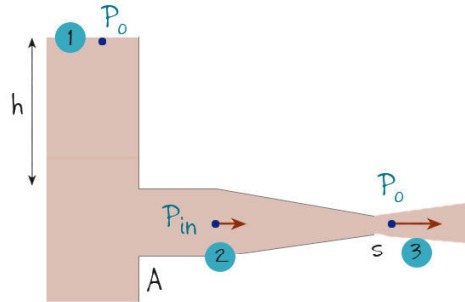
But Momentum of system is NOT zero. As velocity of water at entrance of pipe is different than at exit.

$$\therefore F_{net} = P_{in} A - F - P_o A = \frac{dp}{dt}$$

$$\Rightarrow F = (P_{in} - P_o) A - \frac{dp}{dt}$$

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$$F = (P_{in} - P_0)A - \frac{dp}{dt} \quad (1)$$



Eq 1 Bernoulli between 1 and 3:

$$P_0 + \rho gh = P_0 + \frac{1}{2} \rho v_3^2 \Rightarrow v_3 = \sqrt{2gh}$$

Eq 2 Continuity between 2 and 3:

$$v_2 A = v_3 s \Rightarrow v_2 = v_3 \frac{s}{A} = \sqrt{2gh} \frac{s}{A}$$

Eq 3 Bernoulli between 2 and 3:

$$P_{in} + \frac{1}{2} \rho v_2^2 = P_0 + \frac{1}{2} \rho v_3^2$$

$$\Rightarrow P_{in} - P_0 = \frac{1}{2} \rho (v_3^2 - v_2^2) \quad (2)$$

Eq 4 dp/dt : $\frac{dm v_3 - dm v_2}{dt} = \frac{dm}{dt} (v_3 - v_2) = \rho s v_3 (v_3 - v_2) \quad (3)$

Putting 2 and 3 in 1:

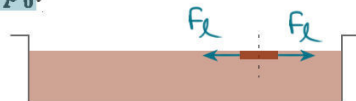
$$F = \frac{1}{2} \rho (v_3^2 - v_2^2) A - \rho s v_3 (v_3 - v_2)$$

$$= \frac{\rho A (2gh) (1 - \frac{s^2}{A^2}) - \rho s (2gh) (1 - \frac{s}{A})}{A} = \frac{\rho gh (A^2 - s^2 - (2As - 2s^2))}{A} = \frac{\rho gh (A - s)^2}{A}$$

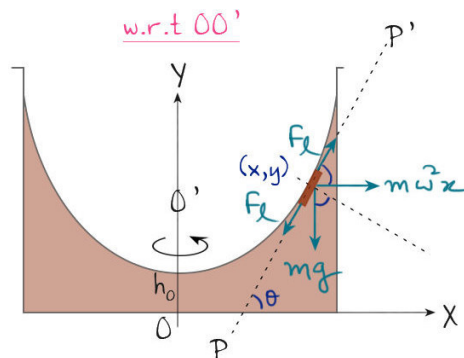
Youtube/Arihant Senior

1.330. A cylindrical vessel with water is rotated about its vertical axis with a constant angular velocity ω . Find:

- the shape of the free surface of the water;
- the water pressure distribution over the bottom of the vessel along its radius provided the pressure at the central point is equal to P_0 .



Force on a surface element by rest of liquid is always normal to surface.
Along the surface, forces are balanced



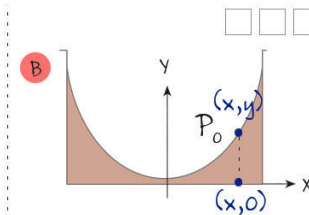
A Lets take an element at (x, y)
Balancing forces along PP' on surface:

$$F_1 + mg \sin \theta = F_2 + m \omega^2 x \cos \theta$$

$$\Rightarrow \tan \theta = \frac{\omega^2 x}{g}$$

$$\frac{dy}{dx} = \frac{\omega^2 x}{g}$$

$$\Rightarrow \int_{h_0}^y dy = \frac{\omega^2}{g} \int_0^x x dx \Rightarrow y = h_0 + \frac{\omega^2 x^2}{2g} \quad (1)$$



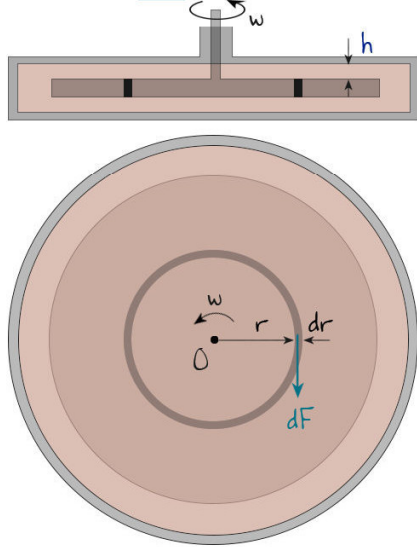
$$P_x = P_0 + \rho g y$$

1 With $h_0 = 0$

$$= P_0 + \rho g \frac{\omega^2 x^2}{2g}$$

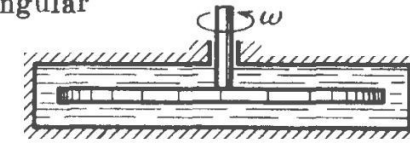
A parabola
Youtube/Arihant Senior

1.331. A thin horizontal disc of radius $R = 10$ cm is located within a cylindrical cavity filled with oil whose viscosity $\eta = 0.08$ P (Fig. 1.89). The clearance between the disc and the horizontal planes of the cavity is equal to $h = 1.0$ mm. Find the power developed by the viscous forces acting on the disc when it rotates with the angular velocity $\omega = 60$ rad/s. The end effects are to be neglected.



Viscous force is given by:

$$F = \eta A \frac{dv}{dx}$$



Tangential Force on elemental ring:

$$dF = \left[\eta (2\pi r dr) \frac{\omega r}{h} \right] \times 2 \quad \text{For top and bottom both}$$

Torque due to dF :

$$dZ = r dF$$

$$\Rightarrow Z = \frac{4\pi\eta\omega}{h} \int_0^R r^3 dr$$

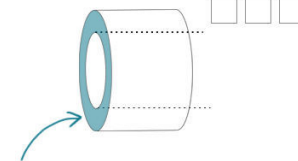
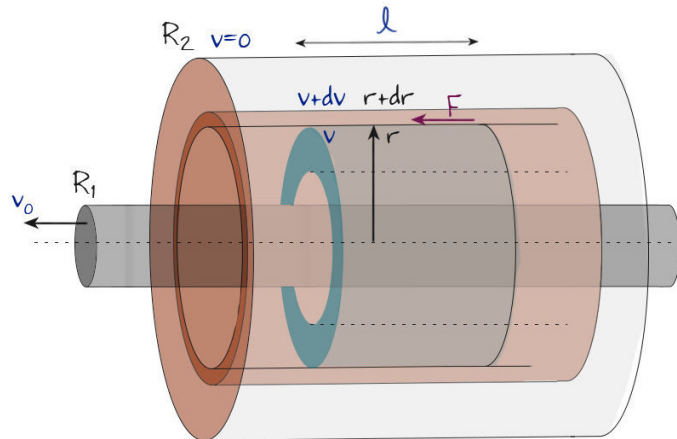
$$\Rightarrow Z = \frac{4\pi\eta\omega}{h} \frac{R^4}{4}$$

$$\therefore \text{Power} = Z\omega$$

$$= \frac{\pi\eta\omega^2 R^4}{h}$$

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1.332. A long cylinder of radius R_1 is displaced along its axis with a constant velocity v_0 inside a stationary co-axial cylinder of radius R_2 . The space between the cylinders is filled with viscous liquid. Find the velocity of the liquid as a function of the distance r from the axis of the cylinders. The flow is laminar.

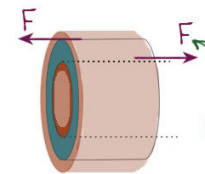


Lets consider a random hollow cylindrical shell of water, with outside radius r , inside radius random and length l .

On outside layer of that shell, Force due to velocity gradient there is given by:

$$F = \eta A \frac{dv}{dr} = \eta (2\pi r l) \frac{dv}{dr}$$

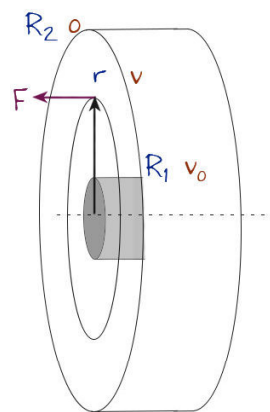
As every particle's velocity is constant, momentum of whole shell is constant too. Therefore net force on shell is zero.



So on inside layer of shell too, force should be F , independent of inside radius.

$\therefore F$ is independent of r , and is a constant.

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F is a constant

$$F = \eta (2\pi r l) \frac{dv}{dr}$$

$$\Rightarrow F \int_{R_1}^{R_2} \frac{dr}{r} = 2\pi \eta l \int_{v_0}^0 dv$$

$$\Rightarrow F \ln \frac{R_2}{R_1} = 2\pi \eta l (-v_0) \quad (1)$$

$$F \int_{R_2}^r \frac{dr}{r} = 2\pi \eta l \int_0^v dv$$

$$\Rightarrow F \ln \frac{r}{R_2} = 2\pi \eta l v \quad (2)$$

Note: We can see that F is negative. It makes sense as actual force on outside layers is retarding and opposite to our assumed direction. Mathematically too it should be negative as dv/dr is negative in $F = \eta A \frac{dv}{dr}$

Dividing 1 and 2:

$$v = \frac{v_0 \ln^2 r / R_2}{\ln R_1 / R_2}$$

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Why there is no pressure difference:

Lets say its Δp per unit length.

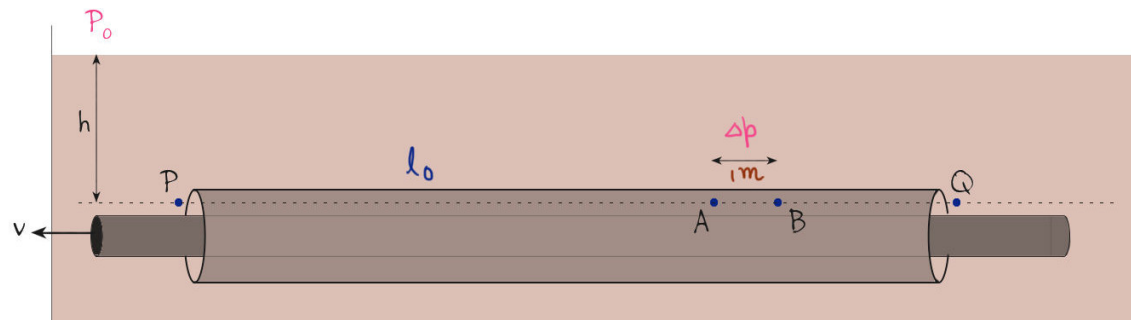
$$\Rightarrow P_A - P_B = \Delta p$$

$$\therefore \text{If length of tube is } l_0, P_P - P_Q = l_0 \Delta p$$

Also,

$$P_P = P_Q = P_0 + \rho g h$$

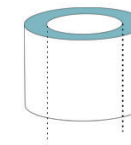
$$\therefore \Delta p = 0$$



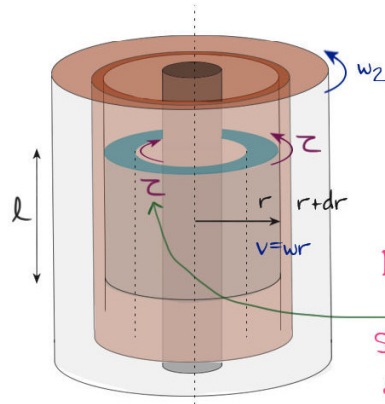
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1.333. A fluid with viscosity η fills the space between two long co-axial cylinders of radii R_1 and R_2 , with $R_1 < R_2$. The inner cylinder is stationary while the outer one is rotated with a constant angular velocity ω_2 . The fluid flow is laminar. Taking into account that the friction force acting on a unit area of a cylindrical surface of radius r is defined by the formula $\sigma = \eta r (\partial\omega/\partial r)$, find:

- (a) the angular velocity of the rotating fluid as a function of radius r ;
 (b) the moment of the friction forces acting on a unit length of the outer cylinder.



Lets consider a hollow cylindrical shell just like in 1.332.



Tangential force on outer layer (of radius r) of shell :

$$F = \eta A \frac{d\omega}{dr} = \eta A \frac{d(\omega r)}{dr} = \eta A r \frac{\partial\omega}{\partial r} \Rightarrow \frac{F}{A} = \eta r \frac{\partial\omega}{\partial r}$$

We derived it

$$\frac{F}{A} = \eta r \frac{\partial\omega}{\partial r}$$

At given r

$$A \rightarrow 2\pi r l$$

Torque due to that force:

$$\tau = F r = \eta A r \frac{\partial\omega}{\partial r} \cdot r = \eta (2\pi r l) r \frac{\partial\omega}{\partial r} r = 2\pi \eta l r^3 \frac{\partial\omega}{\partial r}$$

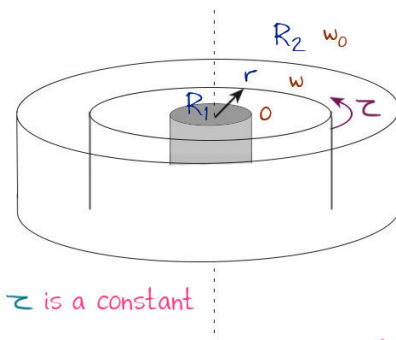
In 1.322, p of each particle was constant, so p of complete shell was constant.

Here: L of each particle's is constant, so L of complete shell is constant.

So on inside layer of shell too, torque should be τ , independent of inside radius.

τ is independent of r , and is a constant.

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τ is a constant

$$\tau = 2\pi \eta l r^3 \frac{d\omega}{dr} \quad (\text{on layer at } r \text{ from center})$$

$$\frac{\tau}{2\pi \eta l} \int_{R_1}^{R_2} \frac{dr}{r^3} = \int_0^{\omega_0} d\omega$$

$$\frac{\tau}{2\pi \eta l} \int_{R_1}^r \frac{dr}{r^3} = \int_0^{\omega} d\omega$$

$$\Rightarrow \frac{\tau}{2\pi \eta l} \frac{1}{2} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right) = \omega_0 \quad (1) \quad \Rightarrow \frac{\tau}{2\pi \eta l} \frac{1}{2} \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right) = \omega \quad (2)$$

A Dividing 2 and 1:

$$\omega = \frac{\omega_0 \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right)}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right)}$$

B From 1

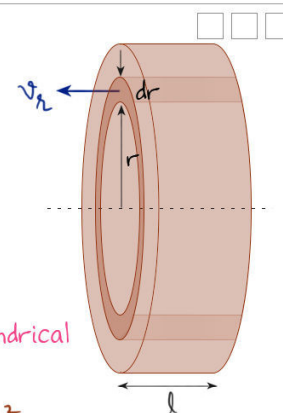
$$\frac{\tau}{l} = \frac{4\pi \eta \omega_0}{\frac{1}{R_1^2} - \frac{1}{R_2^2}}$$

Youtube / Arihant Senior

1.334. A tube of length l and radius R carries a steady flow of fluid whose density is ρ and viscosity η . The fluid flow velocity depends on the distance r from the axis of the tube as $v = v_0(1 - r^2/R^2)$.

Find:

- the volume of the fluid flowing across the section of the tube per unit time;
- the kinetic energy of the fluid within the tube's volume;
- the friction force exerted on the tube by the fluid;
- the pressure difference at the ends of the tube.



A Volume flow rate through a ring element:

$$\begin{aligned}
 v_0(1 - \frac{r^2}{R^2}) & \quad \text{Diagram of ring element with radius } r, \text{ thickness } dr, \text{ and velocity } v_r \\
 d(\frac{dV}{dt}) &= v_r dA \\
 &= v_0(1 - \frac{r^2}{R^2}) 2\pi r dr \\
 \Rightarrow \frac{dV}{dt} &= 2\pi v_0 \int_0^R r(1 - \frac{r^2}{R^2}) dr \\
 &= \frac{\pi v_0 R^2}{2} //
 \end{aligned}$$

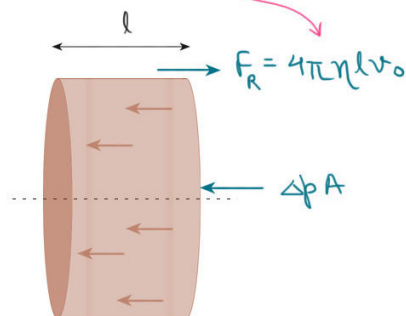
B KE of liquid in cylindrical shell element:

$$\begin{aligned}
 dKE &= \frac{1}{2} dm v_r^2 \\
 &= \frac{1}{2} (\rho 2\pi r dr l) v_r^2 \\
 \Rightarrow KE &= \pi \rho l v_0^2 \int_0^R r(1 - \frac{r^2}{R^2})^2 dr \\
 &= \pi \rho l v_0^2 \frac{R^2}{6} //
 \end{aligned}$$

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$$\begin{aligned}
 \text{C Friction exerted by tube on liquid} &= \eta A \frac{dv}{dr} \Big|_{r=R} \\
 &= \eta (2\pi R l) v_0 (-\frac{2r}{R^2}) \Big|_{r=R} \\
 &= 4\pi \eta l v_0 // \text{ (Independent of R!)}
 \end{aligned}$$

D $\Delta p = ?$ M1



Let pressure difference be Δp

As every particle's velocity is constant, momentum of whole volume is constant too.

$$\therefore F_{net} = 0$$

$$\Rightarrow F_R = \Delta p A$$

$$\Rightarrow \Delta p = \frac{F_R}{A} = \frac{4\pi\eta l v_0}{\pi R^2} = \frac{4\eta l v_0}{R^2} //$$

$$v_r = v_0(1 - \frac{r^2}{R^2})$$

D M2 Poiseuille's equation:

$$\frac{dV}{dt} = \frac{\pi R^4}{8\eta} \frac{\Delta p}{l}$$

From A

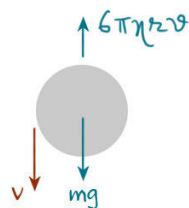
$$\frac{\pi v_0 R^2}{2} = \frac{\pi R^4}{8\eta} \frac{\Delta p}{l}$$

$$\Rightarrow \Delta p = \frac{4\eta l v_0}{R^2} //$$

Youtube/Arihant Senior

1.339. A steel ball of diameter $d = 3.0$ mm starts sinking with zero initial velocity in olive oil whose viscosity is $\eta = 0.90$ P. How soon after the beginning of motion will the velocity of the ball differ from the steady-state velocity by $n = 1.0\%$? $\xi_s \gg \xi_0$

Viscous force = $6\pi\eta r v$



$f = ma$

$\therefore mg - 6\pi\eta r v = m \frac{dv}{dt}$

$\Rightarrow \int_0^{v_{\max}(1-n)} \frac{m dv}{mg - 6\pi\eta r v} = \int_0^t dt$

$\Rightarrow t = \frac{-m}{6\pi\eta r} \ln \frac{mg - 6\pi\eta r v_{\max}(1-n)}{mg}$ (1)

Also, $v_{\max}$ is terminal velocity at which:

$mg = 6\pi\eta r v_{\max}$

$\Rightarrow v_{\max} = \frac{mg}{6\pi\eta r}$ (2)

From 1,2:

$t = \frac{-m}{6\pi\eta r} \ln n$ //

Youtube / Arihant Senior

2.1. A vessel of volume $V = 30$ l contains ideal gas at the temperature 0°C . After a portion of the gas has been let out, the pressure in the vessel decreased by $\Delta p = 0.78$ atm (the temperature remaining constant). Find the mass of the released gas. The gas density under the normal conditions $\rho = 1.3$ g/l.

$P_0 = 1 \text{ atm}$

$T = 0^\circ\text{C}$

$PV = \frac{m}{M} RT_0$

Finally: $\Delta P V = \frac{\Delta m}{M} RT_0$

dividing

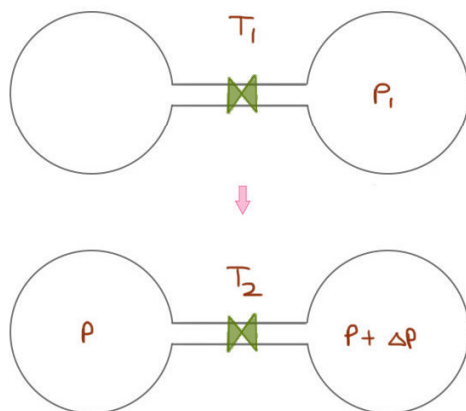
$\frac{\Delta P}{P} = \frac{\Delta m}{m} \Rightarrow$
 $\downarrow P_0 \quad \downarrow m = \rho V$

$\Delta m = \frac{\rho V \Delta P}{P_0}$ //

$\therefore$ At normal conditions: $P \rightarrow P_0$
 $m \rightarrow \rho V$

Youtube / Arihant Senior

2.2. Two identical vessels are connected by a tube with a valve letting the gas pass from one vessel into the other if the pressure difference $\Delta p \geq 1.10 \text{ atm}$. Initially there was a vacuum in one vessel while the other contained ideal gas at a temperature $t_1 = 27^\circ\text{C}$ and pressure $p_1 = 1.00 \text{ atm}$. Then both vessels were heated to a temperature $t_2 = 107^\circ\text{C}$. Up to what value will the pressure in the first vessel (which had vacuum initially) increase?



Conserving moles:

$$\frac{p_1 V}{T_1} = \frac{pV}{T_2} + \frac{(p + \Delta p)V}{T_2}$$

$$\Rightarrow p = \left(\frac{p_1}{T_1} - \frac{\Delta p}{T_2} \right) \frac{T_2}{2}$$

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2.3. A vessel of volume $V = 20 \text{ l}$ contains a mixture of hydrogen and helium at a temperature $t = 20^\circ\text{C}$ and pressure $p = 2.0 \text{ atm}$. The mass of the mixture is equal to $m = 5.0 \text{ g}$. Find the ratio of the mass of hydrogen to that of helium in the given mixture.

Let Hydrogen be n_1 moles.

$$PV = (n_1 + n_2) RT$$

$$n_1 M_1 + n_2 M_2 = m$$

$$n_1 = \frac{\left(m - \frac{PV}{RT} M_2 \right)}{M_1 - M_2}$$

$$n_2 = \frac{\left(m - \frac{PV}{RT} M_1 \right)}{M_2 - M_1}$$

∴ Mass ratio:

$$\frac{m_1}{m_2} = \frac{n_1 M_1}{n_2 M_2}$$

$$= \frac{\left(\frac{PV}{RT} M_2 - m \right) M_1}{\left(m - \frac{PV}{RT} M_1 \right) M_2}$$

$$= \frac{\frac{PV}{RT} - \frac{m}{M_2}}{\frac{m}{M_1} - \frac{PV}{RT}}$$

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2.4. A vessel contains a mixture of nitrogen ($m_1 = 7.0$ g) and carbon dioxide ($m_2 = 11$ g) at a temperature $T = 290$ K and pressure $p_0 = 1.0$ atm. Find the density of this mixture, assuming the gases to be ideal.

$$PV = \left(\frac{m_1}{M_1} + \frac{m_2}{M_2} \right) RT$$

$$\rho = \frac{m_1 + m_2}{V} = \frac{m_1 + m_2}{\left(\frac{m_1}{M_1} + \frac{m_2}{M_2} \right) \frac{P}{RT}} //$$

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2.5. A vessel of volume $V = 7.5$ l contains a mixture of ideal gases at a temperature $T = 300$ K: $v_1 = 0.10$ mole of oxygen, $v_2 = 0.20$ mole of nitrogen, and $v_3 = 0.30$ mole of carbon dioxide. Assuming the gases to be ideal, find:

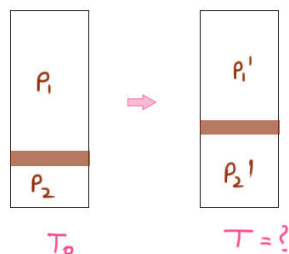
- (a) the pressure of the mixture;
 (b) the mean molar mass M of the given mixture which enters its equation of state $pV = (m/M) RT$, where m is the mass of the mixture.

A
$$P = \frac{(n_1 + n_2 + n_3) RT}{V} //$$

B
$$n_1 + n_2 + n_3 = \frac{m_{\text{net}}}{M_{\text{net}}} = \frac{n_1 M_1 + n_2 M_2 + n_3 M_3}{M_{\text{net}}} \Rightarrow M_{\text{net}} = \frac{n_1 M_1 + n_2 M_2 + n_3 M_3}{n_1 + n_2 + n_3} //$$

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2.6. A vertical cylinder closed from both ends is equipped with an easily moving piston dividing the volume into two parts, each containing one mole of air. In equilibrium at $T_0 = 300 \text{ K}$ the volume of the upper part is $\eta = 4.0$ times greater than that of the lower part. At what temperature will the ratio of these volumes be equal to $\eta' = 3.0$?



Piston is at rest in both cases

$$\therefore mg = (P_2 - P_1)A = (P_2' - P_1')A$$

$$\Rightarrow P_2 - P_1 = P_2' - P_1'$$

$$\Rightarrow \cancel{\eta R} \left(\frac{T_0}{V_2} - \frac{T_0}{V_1} \right) = \cancel{\eta R} \left(\frac{T}{V_2'} - \frac{T}{V_1'} \right)$$

$$\Rightarrow T_0 (\eta + 1) \left(1 - \frac{1}{\eta} \right) = T (\eta' + 1) \left(1 - \frac{1}{\eta'} \right)$$

$$\Rightarrow T = \left(\frac{\eta^2 - 1}{\eta'^2 - 1} \right) \frac{\eta'}{\eta} T_0 //$$

Let total volume be V

$$V_1 \rightarrow \frac{\eta V}{\eta + 1}$$

$$V_2 \rightarrow \frac{V}{\eta + 1}$$

$$V_1 \rightarrow \frac{\eta' V}{\eta' + 1}$$

$$V_2 \rightarrow \frac{V}{\eta' + 1}$$

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2.7. A vessel of volume V is evacuated by means of a piston air pump. One piston stroke captures the volume ΔV . How many strokes are needed to reduce the pressure in the vessel η times? The process is assumed to be isothermal, and the gas ideal.

After 1st stroke: $P_0 V = P_1 (V + \Delta V) \Rightarrow P_1 = P_0 \frac{V}{V + \Delta V}$

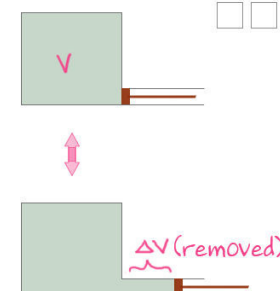
After 2nd stroke: $P_1 V = P_2 (V + \Delta V) \Rightarrow P_2 = P_0 \left(\frac{V}{V + \Delta V} \right)^2$

⋮

After nth stroke:

$$P_n = P_0 \left(\frac{V}{V + \Delta V} \right)^n = \frac{P_0}{\eta} \text{ (given)}$$

$$\Rightarrow n = \frac{\ln \eta}{\ln \frac{V + \Delta V}{V}} //$$



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2.8. Find the pressure of air in a vessel being evacuated as a function of evacuation time t . The vessel volume is V , the initial pressure is p_0 . The process is assumed to be isothermal, and the evacuation rate equal to C and independent of pressure.

Note. The evacuation rate is the gas volume being evacuated per unit time, with that volume being measured under the gas pressure attained by that moment.

At any time t , let the pressure be P .

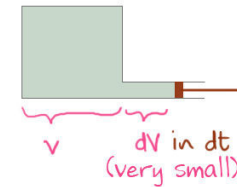
Between t and $t+dt$:

$$PV = (P+dP)(V+dV) \quad \text{where } dV = Cdt$$

$$\Rightarrow PV = PV + PCdt + VdP + C dP dt \quad \text{where } C dP dt \approx 0$$

$$\Rightarrow -\frac{dP}{P} = \frac{C dt}{V}$$

$$\Rightarrow \int_{p_0}^p \frac{dP}{P} = -\frac{C}{V} \int_0^t dt \quad \Rightarrow \quad p = p_0 e^{-Ct/V}$$



$$\Rightarrow \frac{dV}{dt} = C$$

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2.9. A chamber of volume $V = 87 \text{ l}$ is evacuated by a pump whose evacuation rate (see Note to the foregoing problem) equals $C = 10 \text{ l/s}$. How soon will the pressure in the chamber decrease by $\eta = 1000$ times?

$$\text{(from 2.8)} \quad p = p_0 e^{-Ct/V}$$

$$\Rightarrow \frac{p_0}{1000} = p_0 e^{-Ct/V}$$

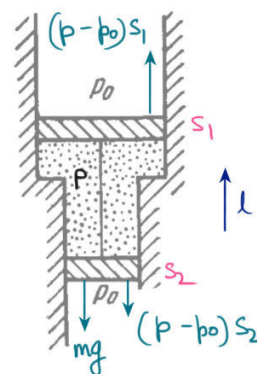
$$\Rightarrow 1000 = e^{\frac{10t}{87}}$$

$$\Rightarrow t = 60 \text{ sec} = 1 \text{ min}$$

(In this realistic situation, we observe that pressure drops quite quickly)

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2.10. A smooth vertical tube having two different sections is open from both ends and equipped with two pistons of different areas (Fig. 2.1). Each piston slides within a respective tube section. One mole of ideal gas is enclosed between the pistons tied with a non-stretchable thread. The cross-sectional area of the upper piston is $\Delta S = 10 \text{ cm}^2$ greater than that of the lower one. The combined mass of the two pistons is equal to $m = 5.0 \text{ kg}$. The outside air pressure is $p_0 = 1.0 \text{ atm}$. By how many kelvins must the gas between the pistons be heated to shift the pistons through $l = 5.0 \text{ cm}$?



When pistons are at rest, $F_{\text{net}} = 0$:

$$\therefore mg + (p - p_0)S_2 = (p - p_0)S_1$$

$$\Rightarrow mg = (p - p_0)\Delta S$$

$$\Rightarrow p = p_0 + \frac{mg}{\Delta S}$$

Let initial Vol = V

$$\text{Final Vol} = V + S_1 l - S_2 l = V + \Delta S l$$

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

$$\Rightarrow \frac{pV}{T} = \frac{p(V + \Delta S l)}{T + \Delta T}$$

$$\Rightarrow 1 + \frac{\Delta T}{T} = 1 + \frac{\Delta S l}{V} \quad \rightarrow \quad \frac{p}{nR} = \frac{T}{V}$$

$$\Rightarrow \Delta T = \Delta S l \left(\frac{T}{V} \right) = \frac{\Delta S l p}{nR} = \frac{\Delta S l (p_0 + \frac{mg}{\Delta S})}{nR}$$

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2.11. Find the maximum attainable temperature of ideal gas in each of the following processes:

(a) $p = p_0 - \alpha V^2$; (b) $p = p_0 e^{-\beta V}$,

where p_0 , α and β are positive constants, and V is the volume of one mole of gas.

$$\text{A } T = \frac{pV}{nR} = \frac{(p_0 - \alpha V^2)V}{nR} \quad (1)$$

$$\text{At } T_{\text{max}} \quad \frac{dT}{dV} = 0 \Rightarrow p_0 - 3\alpha V^2 = 0$$

$$\Rightarrow V = \sqrt{\frac{p_0}{3\alpha}} \quad (2)$$

From 1, 2:

$$\therefore T_{\text{max}} = \frac{\left[p_0 - \alpha \left(\frac{p_0}{3\alpha} \right) \right] \sqrt{\frac{p_0}{3\alpha}}}{nR} = \frac{2p_0}{3\alpha R} \sqrt{\frac{p_0}{3\alpha}}$$

$$\text{B } T = \frac{pV}{nR} = \frac{p_0 e^{-\beta V} V}{nR} \quad (1)$$

$$\text{At } T_{\text{max}} \quad \frac{dT}{dV} = 0 \Rightarrow e^{-\beta V} (1) + V [-\beta e^{-\beta V}] = 0$$

$$\Rightarrow V = 1/\beta \quad (2)$$

From 1, 2:

$$\therefore T_{\text{max}} = \frac{p_0 e^{-1} (1/\beta)}{nR} = \frac{p_0}{e\beta R}$$

2.12. Find the minimum attainable pressure of ideal gas in the process $T = T_0 + \alpha V^2$, where T_0 and α are positive constants, and V is the volume of one mole of gas. Draw the approximate p vs V plot of this process.

$$p = \frac{nRT}{V} = \frac{nR(T_0 + \alpha V^2)}{V} \quad (1)$$

$$\begin{aligned} \text{At } P_{\min} \quad \frac{dp}{dV} = 0 &\Rightarrow V(2\alpha V) - (T_0 + \alpha V^2)(1) = 0 \\ &\Rightarrow V = \sqrt{\frac{T_0}{\alpha}} \quad (2) \end{aligned}$$

From 1,2:

$$\therefore P_{\min} = \frac{nR}{\sqrt{\frac{T_0}{\alpha}}} (T_0 + \alpha \cdot \frac{T_0}{\alpha}) = 2R\sqrt{\alpha T_0} //$$

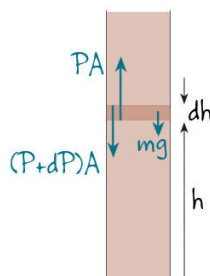
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2.13. A tall cylindrical vessel with gaseous nitrogen is located in a uniform gravitational field in which the free-fall acceleration is equal to g . The temperature of the nitrogen varies along the height h so that its density is the same throughout the volume. Find the temperature gradient dT/dh .

Approach

$$\frac{dT}{dh} = \left(\frac{dT}{dP} \right) \left(\frac{dP}{dh} \right)$$

$P \propto T$ Like water column



Lets say at height h , pressure is P .

On a thin layer then,

$$F_{\text{net}} = 0$$

$$\Rightarrow PA = (P+dP)A + mg \quad \text{where } mg = \rho SA dh$$

$$\Rightarrow -dPA = (\rho SA dh)g$$

$$dP = -\rho g dh$$

(Just like liquid column of density ρ at height h)

$$P = \frac{\rho RT}{M}$$

$$\Rightarrow dT = \frac{M}{\rho R} dP$$

$$\Rightarrow \frac{dT}{dh} = \frac{M}{\rho R} \frac{dP}{dh}$$

$$= \frac{M}{\rho R} \cdot (-\rho g)$$

$$= -\frac{Mg}{R}$$

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2.14. Suppose the pressure p and the density ρ of air are related as $p/\rho^n = \text{const}$ regardless of height (n is a constant here). Find the corresponding temperature gradient.

Approach

$$\frac{dT}{dh} = \frac{dT}{dp} \cdot \frac{dp}{dh}$$

Using $p = \frac{\gamma RT}{M}$ Like water column

Sol: $dp = -\rho g dh$ (From 2.13, also similar to liquid column for small height)

Let, $p = k \rho^n$

Also, $p = \frac{\gamma RT}{M}$

$$\therefore p = \left(\frac{p}{k}\right)^{1/n} \frac{RT}{M}$$

$$\Rightarrow T = \frac{M k^{1/n}}{R} p^{1-1/n}$$

$$\Rightarrow \frac{dT}{dp} = \left(1 - \frac{1}{n}\right) \frac{M k^{1/n}}{R} p^{-1/n}$$

$$= \frac{n-1}{n} \frac{M}{R} \frac{1}{\rho}$$

From A,B

$$\frac{dT}{dh} = -\cancel{\rho} g \cdot \frac{(n-1)M}{nR \cancel{\rho}} = -\frac{(n-1)Mg}{nR} //$$

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2.15. Let us assume that air is under standard conditions close to the Earth's surface. Presuming that the temperature and the molar mass of air are independent of height, find the air pressure at the height 5.0 km over the surface and in a mine at the depth 5.0 km below the surface.

(From 2.13) $dp = -\rho g dh$

Also, $p = \frac{\gamma RT}{M}$

Dividing: $\frac{dp}{p} = -\frac{Mg}{RT} dh$

$$\Rightarrow \int_{p_0}^p \frac{dp}{p} = -\frac{Mg}{RT} \int_0^h dh$$

$$\Rightarrow p = p_0 e^{-\frac{Mgh}{RT}}$$

$h = +5\text{km} \rightarrow p_{+5} = 0.5 \text{ atm} //$

$h = -5\text{km} \rightarrow p_{-5} = 2 \text{ atm} //$

$\therefore$ Pressure increases at faster rate below the surface, than in drops above the surface.

Youtube/Arihant Senior

2.16. Assuming the temperature and the molar mass of air, as well as the free-fall acceleration, to be independent of the height, find the difference in heights at which the air densities at the temperature 0°C differ

(a) e times; (b) by $\eta = 1.0\%$.

(From 2.13) $P = P_0 e^{-\frac{Mgh}{RT}}$

Also, $P \propto \rho$ (At constant T, M)

$$\therefore \rho = \rho_0 e^{-\frac{Mgh}{RT}}$$

$$\Rightarrow h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0}$$

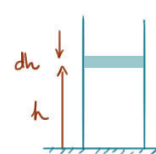
A $\frac{\rho}{\rho_0} = \frac{1}{e} \Rightarrow h = -\frac{RT}{Mg} \ln\left(\frac{1}{e}\right) = 8 \text{ km} //$

B $\frac{\rho}{\rho_0} = 0.99 \Rightarrow h = -\frac{RT}{Mg} \ln 0.99 = 80 \text{ m} //$

$\ln(1 - 0.01) \approx -0.01$

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2.17. An ideal gas of molar mass M is contained in a tall vertical cylindrical vessel whose base area is S and height h . The temperature of the gas is T , its pressure on the bottom base is p_0 . Assuming the temperature and the free-fall acceleration g to be independent of the height, find the mass of gas in the vessel.

dh  $dm = \rho dv$

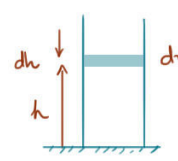
$\rho = \rho_0 e^{-\frac{Mgh}{RT}}$ (From 2.16, at fixed temperature)

$$\Rightarrow m = \int_0^h dm = S \rho_0 \int_0^h e^{-\frac{Mgh}{RT}} dh$$

$$= -S \left(\frac{\rho_0 M}{RT} \right) \frac{RT}{Mg} \left(e^{-\frac{Mgh}{RT}} - 1 \right) = \frac{p_0 S}{g} \left(1 - e^{-\frac{Mgh}{RT}} \right) //$$

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2.18. An ideal gas of molar mass M is contained in a very tall vertical cylindrical vessel in the uniform gravitational field in which the free-fall acceleration equals g . Assuming the gas temperature to be the same and equal to T , find the height at which the centre of gravity of the gas is located.



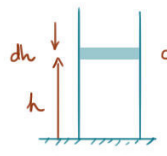
dh
 h
 $dm = S dh$
 $\rho_0 e^{-Mgh/RT}$ (From 2.16, at fixed temperature)
 RT/Mg

$$\Rightarrow h_{cm} = \frac{\int dm h}{\int dm} = \frac{\int_0^\infty \rho_0 S e^{-Mgh/RT} h dh}{\int_0^\infty \rho_0 S e^{-Mgh/RT} dh} = \frac{RT}{Mg}$$

Using: $\int x e^x dx = x e^x - e^x + c$
Youtube/Arihant Senior

2.19. An ideal gas of molar mass M is located in the uniform gravitational field in which the free-fall acceleration is equal to g . Find the gas pressure as a function of height h , if $p = p_0$ at $h = 0$, and the temperature varies with height as

(a) $T = T_0(1 - ah)$; (b) $T = T_0(1 + ah)$, where a is a positive constant.



dh
 h
 $dP = -\rho g dh$
 $\frac{PM}{RT}$
given as $f(h)$

$$\therefore dP = -\frac{\rho M}{RT} g dh$$

$$\frac{dP}{P} = -\frac{Mg}{R} \int \frac{dh}{T}$$

(A) $\int_{p_0}^p \frac{dP}{P} = -\frac{Mg}{R} \int_0^h \frac{dh}{T_0(1 - ah)}$

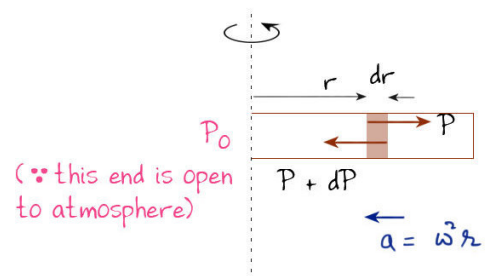
$$\Rightarrow \ln \frac{p}{p_0} = \frac{Mg}{aRT_0} \ln(1 - ah)$$

$$\Rightarrow p = p_0 (1 - ah)^{Mg/aRT_0}$$

replace a with $-a$

(B) $p = p_0 (1 + ah)^{-\frac{Mg}{aRT_0}}$

2.20. A horizontal cylinder closed from one end is rotated with a constant angular velocity ω about a vertical axis passing through the open end of the cylinder. The outside air pressure is equal to p_0 , the temperature to T , and the molar mass of air to M . Find the air pressure as a function of the distance r from the rotation axis. The molar mass is assumed to be independent of r .



(∴ this end is open to atmosphere)

$$F = ma$$

$$\therefore (P + dP)A - PA = \underbrace{dm}_{\substack{\rho dV \\ \downarrow \\ \frac{PM}{RT} A dr}} \cdot \omega^2 r$$

$$\Rightarrow dP \cdot A = \frac{PM}{RT} \cdot A \cdot dr \cdot \omega^2 r$$

$$\Rightarrow \int_{P_0}^P \frac{dP}{P} = \frac{M\omega^2}{RT} \int_0^l r dr \Rightarrow P = P_0 e^{\frac{M\omega^2 r^2}{2RT}}$$

Youtube / Arihant Senior

2.26. Demonstrate that the internal energy U of the air in a room is independent of temperature provided the outside pressure p is constant. Calculate U , if p is equal to the normal atmospheric pressure and the room's volume is equal to $V = 40 \text{ m}^3$.

A Assuming air to be ideal gas, its internal energy is:

$$U = n C_V T$$

$$= \frac{n R}{\gamma - 1} T$$

$$= \frac{PV}{\gamma - 1} = \text{constant, for a room of volume } V \text{ and constant } P. \text{ Hence proved.}$$

B For a given room at 1 atm

$$U = \frac{P_0 V}{\gamma - 1} = \frac{(10^5) (40)}{(1.4 - 1)} = 10^7 \text{ J} \approx \frac{1}{2} 80 (500)^2$$

Kinetic Energy of 80 kg man running at 500 m/s (supersonic) !

Youtube / Arihant Senior

2.27. A thermally insulated vessel containing a gas whose molar mass is equal to M and the ratio of specific heats $C_p/C_v = \gamma$ moves with a velocity v . Find the gas temperature increment resulting from the sudden stoppage of the vessel.

$$U_i + KE_i = U_f + KE_f$$

Due to movement of vessel

$$\therefore \Delta U = KE_i$$

$$\Rightarrow n C_v \Delta T = \frac{1}{2} m v^2$$

$\frac{m}{M} \quad \frac{R}{\gamma-1}$

$$\Rightarrow \Delta T = \frac{\frac{1}{2} m v^2}{\frac{m}{M} \frac{R}{\gamma-1}} = \frac{(\gamma-1) M v^2}{2R} //$$

Youtube / Arihant Senior

2.28. Two thermally insulated vessels 1 and 2 are filled with air and connected by a short tube equipped with a valve. The volumes of the vessels, the pressures and temperatures of air in them are known (V_1, p_1, T_1 and V_2, p_2, T_2). Find the air temperature and pressure established after the opening of the valve.

As vessels are thermally insulated, U is conserved

$$\therefore n_1 C_v T_1 + n_2 C_v T_2 = (n_1 + n_2) C_v T$$

$$\Rightarrow n_1 T_1 + n_2 T_2 = (n_1 + n_2) T$$

$$\Rightarrow T = \frac{n_1 T_1 + n_2 T_2}{n_1 + n_2}$$

$$\Rightarrow T = \frac{p_1 V_1 + p_2 V_2}{\frac{p_1 V_1}{T_1} + \frac{p_2 V_2}{T_2}} //$$

n is conserved

$$\therefore n_1 + n_2 = n_{mix}$$

$$\Rightarrow n_1 + n_2 = \frac{p(V_1 + V_2)}{RT}$$

$$\Rightarrow p = \frac{R(n_1 + n_2)T}{V_1 + V_2}$$

$$= \frac{R(n_1 T_1 + n_2 T_2)}{V_1 + V_2}$$

$$= \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2} //$$

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2.29. Gaseous hydrogen contained initially under standard conditions in a sealed vessel of volume $V = 5.0 \text{ l}$ was cooled by $\Delta T = 55 \text{ K}$. Find how much the internal energy of the gas will change and what amount of heat will be lost by the gas.

$$\text{A } \Delta U = n c_v \Delta T = \frac{5}{22.4} \cdot \frac{5 \cdot (8.3)}{2} \cdot (-55) = -0.25 \text{ kJ} //$$

$$\text{B } Q = \Delta U + \int P dV \Rightarrow Q = \Delta U = -0.25 \text{ kJ}$$

Heat lost: $-Q = 0.25 \text{ kJ} //$

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2.30. What amount of heat is to be transferred to nitrogen in the isobaric heating process for that gas to perform the work $A = 2.0 \text{ J}$?

$$\begin{aligned} Q &= \Delta U + W = \frac{W}{\gamma-1} + W = \frac{\gamma}{\gamma-1} W = \frac{1.4}{(1.4-1)} \cdot 2 = 7 \text{ J} // \\ &= n c_v \Delta T \\ &= \frac{n R \Delta T}{\gamma-1} \\ &= \frac{P \Delta V}{\gamma-1} \\ &= \frac{W}{\gamma-1} \end{aligned}$$

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2.31. As a result of the isobaric heating by $\Delta T = 72 \text{ K}$ one mole of a certain ideal gas obtains an amount of heat $Q = 1.60 \text{ kJ}$. Find the work performed by the gas, the increment of its internal energy, and the value of $\gamma = C_p/C_v$.

A $W = p \Delta V = \gamma R \Delta T = (8.3)(72) = 0.6 \text{ kJ} //$

B $\Delta U = Q - W = 1.6 - 0.6 = 1 \text{ kJ} //$

C $\Delta U = \frac{\gamma R \Delta T}{\gamma - 1} = 1 \text{ kJ}$

$\Rightarrow \gamma = 1.6 //$

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2.32. Two moles of a certain ideal gas at a temperature $T_0 = 300 \text{ K}$ were cooled isochorically so that the gas pressure reduced $\eta = 2.0$ times. Then, as a result of the isobaric process, the gas expanded till its temperature got back to the initial value. Find the total amount of heat absorbed by the gas in this process.

In phase 1: $V = \text{const}$

$$\frac{P_0}{T_0} = \frac{P_1}{T_1} \Rightarrow T_1 = T_0 \left(\frac{P_1}{P_0} \right) = \frac{T_0}{\eta}$$

i.e, Temperature got reduced η times.

$$\begin{aligned} \text{Also, } Q_1 &= \Delta U \\ &= \frac{n R \Delta T}{\gamma - 1} \\ &= \frac{n R \left(\frac{T_0}{\eta} - T_0 \right)}{\gamma - 1} \end{aligned}$$

In phase 2: $P = \text{constant}$

$$\begin{aligned} Q_2 &= \Delta U + p \Delta V \\ &= \frac{n R \Delta T}{\gamma - 1} + n R \Delta T \\ &= \frac{\gamma}{\gamma - 1} n R \left(T_0 - \frac{T_0}{\eta} \right) \end{aligned}$$

$$\begin{aligned} \therefore \text{Net Heat: } Q &= Q_1 + Q_2 \\ &= n R T_0 \left(\frac{\gamma - 1}{\eta} \right) \quad (\text{Independent of } \gamma) \end{aligned}$$

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2.33. Calculate the value of $\gamma = C_p/C_v$ for a gaseous mixture consisting of $v_1 = 2.0$ moles of oxygen and $v_2 = 3.0$ moles of carbon dioxide. The gases are assumed to be ideal.

Method 1

$$U_{\text{mix}} = U_1 + U_2$$

$$(n_1 + n_2) \frac{R}{(\gamma - 1)} T = \frac{n_1 R T}{\gamma_1 - 1} + \frac{n_2 R T}{\gamma_2 - 1}$$

$$\Rightarrow \gamma = \frac{n_1 \gamma_1 (\gamma_2 - 1) + n_2 \gamma_2 (\gamma_1 - 1)}{n_1 (\gamma_2 - 1) + n_2 (\gamma_1 - 1)}$$

M2

$$C_v = \frac{n_1 C_{v1} + n_2 C_{v2}}{n}$$

$$= \frac{n_1 R}{\gamma_1 - 1} + \frac{n_2 R}{\gamma_2 - 1}$$

$$C_p = \frac{n_1 C_{p1} + n_2 C_{p2}}{n}$$

$$= \frac{n_1 \gamma_1 R}{\gamma_1 - 1} + \frac{n_2 \gamma_2 R}{\gamma_2 - 1}$$

$$\therefore \gamma = \frac{C_p}{C_v} = \frac{\frac{n_1 \gamma_1}{\gamma_1 - 1} + \frac{n_2 \gamma_2}{\gamma_2 - 1}}{\frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}}$$

$$= \frac{n_1 \gamma_1 (\gamma_2 - 1) + n_2 \gamma_2 (\gamma_1 - 1)}{n_1 (\gamma_2 - 1) + n_2 (\gamma_1 - 1)}$$

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2.34. Find the specific heat capacities c_v and c_p for a gaseous mixture consisting of 7.0 g of nitrogen and 20 g of argon. The gases are assumed to be ideal.

$$C_v = \frac{\frac{n_1 R}{\gamma_1 - 1} + \frac{n_2 R}{\gamma_2 - 1}}{n_1 + n_2} = 15.2 \text{ J/mol-K}$$

$$C_p = \frac{\frac{n_1 \gamma_1 R}{\gamma_1 - 1} + \frac{n_2 \gamma_2 R}{\gamma_2 - 1}}{n_1 + n_2} = 23.8 \text{ J/mol-K}$$

or
 $C_p = C_v + R$

Specific heat capacities:

$$C_v' = \frac{C_v}{M_{\text{mix}}} = 0.42 \text{ J/g-K}$$

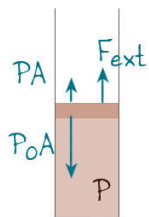
$$C_p' = \frac{C_p}{M_{\text{mix}}} = 0.66 \text{ J/g-K}$$

$$M_{\text{mix}} = \frac{m_1 + m_2}{n_1 + n_2} = \frac{7 + 20}{\frac{7}{28} + \frac{20}{40}} = \frac{27}{\frac{1}{4} + \frac{1}{2}} = 36$$

• diatomic

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2.35. One mole of a certain ideal gas is contained under a weightless piston of a vertical cylinder at a temperature T . The space over the piston opens into the atmosphere. What work has to be performed in order to increase isothermally the gas volume under the piston n times by slowly raising the piston? The friction of the piston against the cylinder walls is negligibly small.



$$dW_{\text{ext}} = F_{\text{ext}} dx = (P_0 - P) A dx$$

$$= (P_0 - P) dV$$

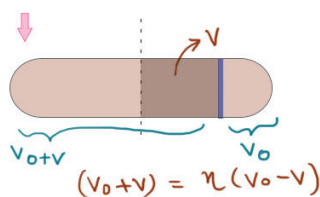
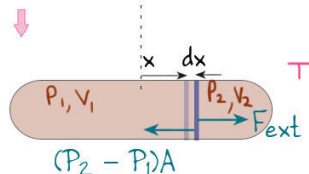
$$\Rightarrow W_{\text{ext}} = \int_{V_0}^{\eta V_0} (P_0 - P) dV$$

$$= (\eta - 1) P_0 V_0 - nRT_0 \int_{V_0}^{\eta V_0} \frac{dV}{V}$$

$$= (\eta - 1) nRT_0 - nRT_0 \ln \eta = nRT_0 (\eta - 1 - \ln \eta)$$

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2.36. A piston can freely move inside a horizontal cylinder closed from both ends. Initially, the piston separates the inside space of the cylinder into two equal parts each of volume V_0 , in which an ideal gas is contained under the same pressure p_0 and at the same temperature. What work has to be performed in order to increase isothermally the volume of one part of gas η times compared to that of the other by slowly moving the piston?



$$dW = F_{\text{ext}} dx$$

$$= (P_2 - P_1) A dx$$

$$= \left(\frac{P_0 V_0}{V_0 - Ax} - \frac{P_0 V_0}{V_0 + Ax} \right) A dx$$

$$= \frac{(P_0 V_0) 2Ax \cdot A dx}{V_0^2 - A^2 x^2}$$

$\begin{cases} Ax \rightarrow V \\ A dx \rightarrow dV \end{cases}$

$$\therefore W = 2P_0 V_0 \int_0^V \frac{V dV}{V_0^2 - V^2}$$

$$= P_0 V_0 \ln \frac{(\eta + 1)^2}{4\eta}$$

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2.37. Three moles of an ideal gas being initially at a temperature $T_0 = 273 \text{ K}$ were isothermally expanded $n = 5.0$ times its initial volume and then isochorically heated so that the pressure in the final state became equal to that in the initial state. The total amount of heat transferred to the gas during the process equals $Q = 80 \text{ kJ}$. Find the ratio $\gamma = C_p/C_v$ for this gas.

$$P_0, V, T_0$$

↓ Isothermal

$$\frac{P_0}{n}, nV, T_0$$

↓ Isochoric

$$P_0, nV, nT_0$$

∴ In isothermal expansion:

$$Q_1 = nRT_0 \ln \frac{V_2}{V_1} = nRT_0 \ln n$$

And, in isochoric process:

$$Q_2 = nC_v \Delta T = \frac{nR}{\gamma-1} (n-1)T_0$$

$$Q = Q_1 + Q_2$$

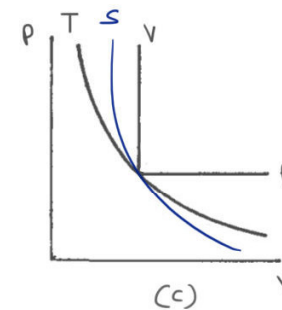
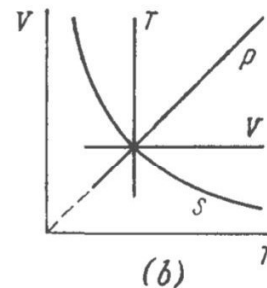
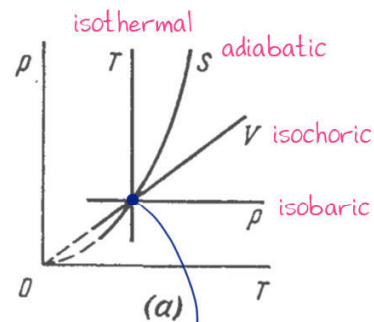
$$= nRT_0 \left(\ln n + \frac{n-1}{\gamma-1} \right)$$

$$\Rightarrow \gamma = 1 + \frac{n-1}{\frac{Q}{nRT_0} - \ln n} //$$

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2.38. Draw the approximate plots of isochoric, isobaric, isothermal, and adiabatic processes for the case of an ideal gas, using the following variables:

(a) P, T ; (b) V, T . (c) P, V



Slope of adiabatic is sharper than isothermal in PV graph.

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2.39. One mole of oxygen being initially at a temperature $T_0 = 290 \text{ K}$ is adiabatically compressed to increase its pressure $\eta = 10.0$ times. Find:

- (a) the gas temperature after the compression;
(b) the work that has been performed on the gas.

$$PV^\gamma = \text{const}$$

Sol: A For given adiabatic process

$$\Rightarrow P \left(\frac{T}{P} \right)^\gamma = \text{const}$$

$$\Rightarrow P^{1-\gamma} T^\gamma = \text{const}$$

$$\Rightarrow P^{\frac{1-\gamma}{\gamma}} T = \text{const}$$

$$P^{\frac{1-\gamma}{\gamma}} T_0 = (\eta P)^{\frac{1-\gamma}{\gamma}} T_f$$

$$\Rightarrow T_f = \frac{T_0}{\eta^{\frac{1-\gamma}{\gamma}}} = T_0 \eta^{\frac{\gamma-1}{\gamma}}$$

$$\begin{aligned} \text{B Work done on the gas} &= \Delta U \\ &= \frac{\eta R \Delta T}{\gamma - 1} = \frac{\eta R T_0}{\gamma - 1} \left(\eta^{\frac{\gamma-1}{\gamma}} - 1 \right) \end{aligned}$$

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2.40. A certain mass of nitrogen was compressed $\eta = 5.0$ times (in terms of volume), first adiabatically, and then isothermally. In both cases the initial state of the gas was the same. Find the ratio of the respective works expended in each compression.

External work

Work done by the gas:

$$W_{\text{adi}} = \frac{-nR\Delta T}{\gamma - 1} = \frac{-nRT_0(\eta^{\frac{\gamma-1}{\gamma}} - 1)}{\gamma - 1}$$

$$W_{\text{iso}} = nRT_0 \ln \frac{V_2}{V_1} = nRT_0 \ln \frac{1}{\eta}$$

$$\begin{aligned} \therefore \text{Ratio of work done on the gas} &= \frac{-W_{\text{adi}}}{-W_{\text{iso}}} \\ &= \frac{(\eta^{\frac{\gamma-1}{\gamma}} - 1)}{(\gamma - 1) \ln \eta} \end{aligned}$$

$$PV^\gamma = \text{const}$$

$$\Rightarrow \left(\frac{T}{V} \right)^\gamma = \text{const}$$

$$\Rightarrow T V^{\gamma-1} = \text{const}$$

$$\therefore \text{as given: } T_0 V^{\gamma-1} = T_f \left(\frac{V}{\eta} \right)^{\gamma-1}$$

$$\Rightarrow T_f = T_0 \eta^{\gamma-1}$$

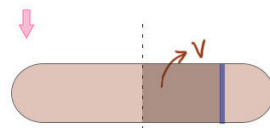
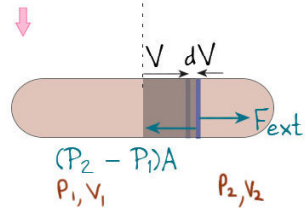
$$\Rightarrow \Delta T = T_f - T_0 = T_0 (\eta^{\gamma-1} - 1)$$

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2.41. A heat-conducting piston can freely move inside a closed thermally insulated cylinder with an ideal gas. In equilibrium the piston divides the cylinder into two equal parts, the gas temperature being equal to T_0 . The piston is slowly displaced. Find the gas temperature as a function of the ratio η of the volumes of the greater and smaller sections. The adiabatic exponent of the gas is equal to γ .

concept: $W_{\text{ext}} = \Delta U$

$\Rightarrow dW_{\text{ext}} = dU$



$$dW = F_{\text{ext}} dx = (p_2 - p_1) A dx$$

$$= (p_2 - p_1) dV$$

$$= \left(\frac{nRT}{V_2} - \frac{nRT}{V_1} \right) dV$$

$$= nRT \left(\frac{1}{V_0 - V} - \frac{1}{V_0 + V} \right) dV$$

$$= 2nRT \frac{V dV}{V_0^2 - V^2} = dU = \frac{2nR}{\gamma - 1} dT$$

$$(V_0 + V) = \eta (V_0 - V) \Rightarrow V = \left(\frac{\eta - 1}{\eta + 1} \right) V_0 \Rightarrow \int_0^{V_0} \frac{V dV}{V_0^2 - V^2} = \left(\frac{\eta - 1}{\eta + 1} \right) \int_{T_0}^T \frac{dT}{T} \Rightarrow T = T_0 \left[\frac{(\eta + 1)^2}{4\eta} \right]^{\frac{\gamma - 1}{2}}$$

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2.43. The volume of one mole of an ideal gas with the adiabatic exponent γ is varied according to the law $V = a/T$, where a is a constant. Find the amount of heat obtained by the gas in this process if the gas temperature increased by ΔT .

$$\Delta U = \frac{nR\Delta T}{\gamma - 1}$$

$$W = \int p dV$$

$$= \int \frac{nRT}{V} dV$$

$$= \int \frac{nRT}{\left(\frac{a}{T} \right)} \left(-\frac{a}{T^2} dT \right)$$

$$= -nR \int_T^{T+\Delta T} dT$$

$$= -nR\Delta T$$

$$Q = \Delta U + W$$

$$= nR\Delta T \left(\frac{1}{\gamma - 1} - 1 \right)$$

$$= nR\Delta T \cdot \frac{2 - \gamma}{\gamma - 1}$$

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2.44. Demonstrate that the process in which the work performed by an ideal gas is proportional to the corresponding increment of its internal energy is described by the equation $pV^n = \text{const}$, where n is a constant.

$$dW = PdV = \frac{nRT}{V} dV \quad (1)$$

$$dU = \frac{nRdT}{\gamma-1} \quad (2)$$

$$W \propto U \text{ (given)}$$

$$\Rightarrow dW \propto dU$$

$$\Rightarrow dW = k_1 dU$$

$$\cancel{n}RT \frac{dV}{V} = k_1 \frac{\cancel{n}RdT}{\gamma-1}$$

$$\Rightarrow \int \frac{dV}{V} = k_2 \int \frac{dT}{T}$$

$$\Rightarrow \ln V = k_2 \ln T + k_3 \quad \xrightarrow{k_2 \ln k_4}$$

$$\Rightarrow \ln V = k_2 \ln(k_4 T)$$

$$\Rightarrow V = k_4 T^{k_2}$$

$$\Rightarrow V = k_4 \left(\frac{pV}{nR} \right)^{k_2}$$

$$\Rightarrow V^{1/k_2} = k_4^{1/k_2} \cdot \frac{pV}{nR}$$

$$\Rightarrow p = k_5 V^{\frac{1}{k_2}-1}$$

$$\Rightarrow p V^{1-\frac{1}{k_2}} = k_5$$

$$\Rightarrow pV^n = \text{constant}$$

Hence proved

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2.45. Find the molar heat capacity of an ideal gas in a polytropic process $pV^n = \text{const}$ if the adiabatic exponent of the gas is equal to γ . At what values of the polytropic constant n will the heat capacity of the gas be negative?

A For a polytropic process of, $pV^n = \text{const}$

$$W = \frac{n_0 R (T_1 - T_2)}{n-1}$$

$$\text{Also, } \Delta U = \frac{n_0 R (T_2 - T_1)}{\gamma-1}$$

$$\therefore Q = \Delta U + W$$

$$\Rightarrow n_0 C (T_2 - T_1) = \frac{n_0 R (T_2 - T_1)}{\gamma-1} + \frac{-n_0 R (T_2 - T_1)}{n-1}$$

$$\Rightarrow C = R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right] = \frac{R (n-\gamma)}{(\gamma-1)(n-1)} \quad (1)$$

B From 1:

$$C < 0, \text{ if } \frac{n-\gamma}{n-1} < 0$$

$$\Rightarrow 1 < n < \gamma //$$

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2.46. In a certain polytropic process the volume of argon was increased $\alpha = 4.0$ times. Simultaneously, the pressure decreased $\beta = 8.0$ times. Find the molar heat capacity of argon in this process, assuming the gas to be ideal.

$$V \rightarrow \alpha V$$

$$P \rightarrow P/\beta$$

Let the process be:

$$PV^n = \text{constant}$$

$$\Rightarrow P V^n = \left(\frac{P}{\beta}\right) (\alpha V)^n$$

$$\Rightarrow n = \frac{\ln \beta}{\ln \alpha}$$

We derived in 2.45, $C = R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right]$

$$\therefore C = R \left[\frac{1}{\gamma-1} - \frac{\ln \alpha}{\ln \beta - \ln \alpha} \right] //$$

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2.47. One mole of argon is expanded polytropically, the polytropic constant being $n = 1.50$. In the process, the gas temperature changes by $\Delta T = -26$ K. Find:

- (a) the amount of heat obtained by the gas;
(b) the work performed by the gas.

Sol: A $Q = n_0 C \Delta T$
 $= n_0 R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right] \Delta T //$

B $W = - \frac{n_0 R \Delta T}{n-1} //$

For a polytropic process, $PV^n = \text{const}$

$$C = R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right] \text{ (Derived in 2.45.)}$$

$$W = - \frac{n_0 R \Delta T}{n-1}$$

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2.48. An ideal gas whose adiabatic exponent equals γ is expanded according to the law $p = \alpha V$, where α is a constant. The initial volume of the gas is equal to V_0 . As a result of expansion the volume increases η times. Find:

- the increment of the internal energy of the gas;
- the work performed by the gas;
- the molar heat capacity of the gas in the process.

Polytropic process: $pV^{-1} = \alpha \quad \equiv \quad pV^\eta = \text{const}$

Finding ΔT

$$\left(\frac{nRT}{V}\right)V^{-1} = \alpha$$

$$\Rightarrow \frac{T_0}{V_0^2} = \frac{T_2}{(\eta V_0)^2}$$

$$\Rightarrow T_2 = \eta^2 T_0$$

$$\Delta T = T_2 - T_0 = (\eta^2 - 1) T_0 = (\eta^2 - 1) \frac{p_0 V_0}{n_0 R} = \frac{(\eta^2 - 1) \alpha V_0^2}{n_0 R}$$

For $pV^\eta = \text{const}$

$$C = R \left[\frac{1}{\gamma - 1} - \frac{1}{\eta - 1} \right]$$

$$W = - \frac{n_0 R \Delta T}{\eta - 1}$$

A $\Delta U = \frac{n_0 R \Delta T}{\gamma - 1} = \frac{(\eta^2 - 1) \alpha V_0^2}{\gamma - 1}$

B $W = - \frac{n_0 R \Delta T}{\eta - 1} = - \frac{(\eta^2 - 1) \alpha V_0^2}{\eta - 1}$

C $C = R \left[\frac{1}{\gamma - 1} - \frac{1}{\eta - 1} \right]$

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2.49. An ideal gas whose adiabatic exponent equals γ is expanded so that the amount of heat transferred to the gas is equal to the decrease of its internal energy. Find:

- the molar heat capacity of the gas in this process;
- the equation of the process in the variables T, V ;
- the work performed by one mole of the gas when its volume increases η times if the initial temperature of the gas is T_0 .

For $pV^\eta = \text{const}$

1 $C = R \left[\frac{1}{\gamma - 1} - \frac{1}{\eta - 1} \right]$

2 $W = - \frac{n_0 R \Delta T}{\eta - 1}$

Heat = Decrease in U

$$\therefore n_0 C dT = - n_0 C_V dT$$

$$C = - \frac{R}{\gamma - 1} = R \left[\frac{1}{\gamma - 1} - \frac{1}{\eta - 1} \right]$$

$$\Rightarrow 2(\eta - 1) = \gamma - 1$$

$$\Rightarrow \eta = \frac{\gamma + 1}{2} \quad 3$$

Process is: $pV^{\frac{\gamma+1}{2}} = \text{const} \quad 4$

A $C = R \left[\frac{1}{\gamma - 1} - \frac{1}{\eta - 1} \right] = - \frac{R}{\gamma - 1}$

B $pV^{\frac{\gamma+1}{2}} = \text{const} \Rightarrow \left(\frac{T}{V}\right)V^{\frac{\gamma+1}{2}} = \text{const} \Rightarrow T V^{\frac{\gamma-1}{2}} = \text{const}$

C $\left(\frac{T}{V}\right)V^{\frac{\gamma+1}{2}} = \text{const} \Rightarrow T_0 V^{\frac{\gamma-1}{2}} = T_2 (\eta V)^{\frac{\gamma-1}{2}}$

$$\Rightarrow T_2 = T_0 \eta^{\frac{1-\gamma}{2}} \Rightarrow \Delta T = T_0 \left(\eta^{\frac{1-\gamma}{2}} - 1 \right)$$

$$\therefore W = - \frac{n_0 R \Delta T}{\eta - 1} = - \frac{n_0 R T_0 \left(\eta^{\frac{1-\gamma}{2}} - 1 \right)}{\frac{\gamma+1}{2} - 1} = \frac{2 n_0 R T_0 (1 - \eta^{\frac{1-\gamma}{2}})}{\gamma - 1}$$

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2.50. One mole of an ideal gas whose adiabatic exponent equals γ undergoes a process in which the gas pressure relates to the temperature as $p = aT^\alpha$, where a and α are constants. Find:

(a) the work performed by the gas if its temperature gets an increment ΔT ;

(b) the molar heat capacity of the gas in this process; at what value of α will the heat capacity be negative?

For $PV^\eta = \text{const}$

$$1 \quad C = R \left[\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right]$$

$$2 \quad W = - \frac{n_0 R \Delta T}{\alpha-1}$$

$$P = a T^\alpha$$

$$\Rightarrow P = a \left(\frac{PV}{nR} \right)^\alpha$$

$$A \quad W = - \frac{n_0 R \Delta T}{\alpha-1} = - \frac{n_0 R \Delta T}{\frac{\alpha}{\alpha-1} - 1} = (1-\alpha) n_0 R \Delta T //$$

∴ Process is:

$$PV^{\frac{\alpha}{\alpha-1}} = \text{const}$$

$$B \quad C = \frac{R}{\gamma-1} - \frac{R}{\frac{\alpha}{\alpha-1} - 1} = \frac{R}{\gamma-1} - (\alpha-1) R //$$

$$\Rightarrow n = \frac{\alpha}{\alpha-1}$$

$$C < 0 \text{ if } \frac{1}{\gamma-1} < \alpha-1 \Rightarrow \alpha > \frac{\gamma}{\gamma-1} //$$

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2.51. An ideal gas with the adiabatic exponent γ undergoes a process in which its internal energy relates to the volume as $U = aV^\alpha$, where a and α are constants. Find:

(a) the work performed by the gas and the amount of heat to be transferred to this gas to increase its internal energy by ΔU ;

(b) the molar heat capacity of the gas in this process.

For $PV^\eta = \text{const}$

$$C = R \left[\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right]$$

$$W = - \frac{n_0 R \Delta T}{\alpha-1}$$

$$U = a V^\alpha$$

$$\downarrow$$

$$\frac{nRT}{\gamma-1} = a V^\alpha$$

$$\frac{PV}{\gamma-1} = a V^\alpha$$

∴ Process is:

$$PV^{1-\alpha} = \text{const}$$

$$n = 1-\alpha$$

$$A \quad W = - \frac{nR\Delta T}{\alpha-1} = - \left(\frac{nR\Delta T}{\gamma-1} \right) \frac{(\gamma-1)}{\alpha-1}$$

$$= \frac{\Delta U (1-\gamma)}{\alpha-1}$$

$$= \frac{\Delta U (1-\gamma)}{(1-\alpha-1)} = \frac{(\gamma-1) \Delta U}{\alpha} //$$

$$B \quad C = R \left[\frac{1}{\gamma-1} - \frac{1}{-\alpha} \right] //$$

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2.52. An ideal gas has a molar heat capacity C_V at constant volume. Find the molar heat capacity of this gas as a function of its volume V , if the gas undergoes the following process:

(a) $T = T_0 e^{\alpha V}$; (b) $p = p_0 e^{\alpha V}$, These are NOT polytropic ($pV^\gamma = \text{const}$) processes. where T_0 , p_0 , and α are constants.

$$dQ = dU + dW$$

$$\Rightarrow n_0 C dT = n_0 C_V dT + p dV$$

$$\Rightarrow C = C_V + \frac{p dV}{n_0 dT}$$

A $T = T_0 e^{\alpha V}$

$$\frac{dT}{dV} = T_0 \alpha e^{\alpha V}$$

$$\therefore C = C_V + \left(\frac{n_0 R T}{V} \right) \frac{dV}{n_0 dT} = C_V + \frac{R T_0 e^{\alpha V}}{V T_0 \alpha e^{\alpha V}} = C_V + \frac{R}{\alpha V} //$$

B $p = p_0 e^{\alpha V}$

$$n_0 R T = (p_0 e^{\alpha V}) V$$

$$n_0 R \frac{dT}{dV} = p_0 [e^{\alpha V} + V \alpha e^{\alpha V}] = p_0 e^{\alpha V} (1 + \alpha V)$$

$$\therefore C = C_V + \frac{p_0 e^{\alpha V} R}{p_0 e^{\alpha V} (1 + \alpha V)} = C_V + \frac{R}{1 + \alpha V} //$$

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2.53. One mole of an ideal gas whose adiabatic exponent equals γ undergoes a process $p = p_0 + \alpha/V$, where p_0 and α are positive constants. Find:

(a) heat capacity of the gas as a function of its volume;

(b) the internal energy increment of the gas, the work performed by it, and the amount of heat transferred to the gas, if its volume increased from V_1 to V_2 .

$$dQ = dU + dW$$

$$\Rightarrow n_0 C dT = \frac{n_0 R dT}{\gamma - 1} + p dV$$

$$\Rightarrow C = \frac{R}{\gamma - 1} + \frac{p dV}{n_0 dT}$$

A $p = p_0 + \frac{\alpha}{V}$

$$\Rightarrow n_0 R T = (p_0 + \frac{\alpha}{V}) V$$

$$\Rightarrow n_0 R dT = p_0 dV$$

$$\therefore C = \frac{R}{\gamma - 1} + \left(p_0 + \frac{\alpha}{V} \right) \frac{R}{p_0}$$

$$= \frac{R}{\gamma - 1} + R \left(1 + \frac{\alpha}{p_0 V} \right) //$$

Finding, ΔT

$$p = p_0 + \frac{\alpha}{V}$$

$$n_0 R T = p_0 V + \alpha$$

$$n_0 R \Delta T = p_0 \Delta V$$

$$\Delta T = \frac{p_0 (V_2 - V_1)}{n_0 R}$$

B $\Delta U = n_0 C_V \Delta T = \frac{p_0 (V_2 - V_1)}{\gamma - 1}$

$$W = \int p dV = \int_{V_1}^{V_2} \left(p_0 + \frac{\alpha}{V} \right) dV$$

$$= p_0 (V_2 - V_1) + \alpha \ln V_2 / V_1 //$$

$$Q = \Delta U + W //$$

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2.54. One mole of an ideal gas with heat capacity at constant pressure C_p undergoes the process $T = T_0 + \alpha V$, where T_0 and α are constants. Find:

- (a) heat capacity of the gas as a function of its volume;
 (b) the amount of heat transferred to the gas, if its volume increased from V_1 to V_2 .

A $n_0 C dT = n_0 C_v dT + P dV$

$$C = C_p - R + \frac{P dV}{n_0 dT} \rightarrow \frac{dT}{dV} = \frac{d(T_0 + \alpha V)}{dV} = \alpha$$

$$\Rightarrow C = C_p - R + \frac{n_0 R (T_0 + \alpha V)}{V n_0 \alpha}$$

$$= C_p - R + R \left(\frac{T_0}{\alpha V} + 1 \right)$$

$$= C_p + \frac{RT_0}{\alpha V} //$$

B $dQ = n_0 C dT$

$$dQ = n_0 \left(C_p + \frac{RT_0}{\alpha V} \right) \alpha dV$$

$$Q = n_0 \alpha \int_{V_1}^{V_2} \left(C_p + \frac{RT_0}{\alpha V} \right) dV$$

$$= n_0 C_p \alpha (V_2 - V_1) + n_0 R T_0 \ln \frac{V_2}{V_1} //$$

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2.55. For the case of an ideal gas find the equation of the process (in the variables T, V) in which the molar heat capacity varies as:

- (a) $C = C_v + \alpha T$; (b) $C = C_v + \beta V$; (c) $C = C_v + aP$, where α, β , and a are constants.

$$n_0 C dT = n_0 C_v dT + P dV$$

$$\Rightarrow C = C_v + \frac{P dV}{n_0 dT}$$

A $\frac{P dV}{n_0 dT} = \alpha T$

$$\Rightarrow \left(\frac{n_0 R T}{V} \right) \frac{dV}{n_0 dT} = \alpha T$$

$$\Rightarrow R \int \frac{dV}{V} = \alpha \int dT$$

$$\Rightarrow R \ln V = \alpha T + k //$$

B $\frac{P dV}{n_0 dT} = \beta V$

$$\Rightarrow \left(\frac{n_0 R T}{V} \right) \frac{dV}{n_0 dT} = \beta V$$

$$\Rightarrow R \int \frac{dV}{V^2} = \beta \int \frac{dT}{T}$$

$$\Rightarrow -\frac{R}{\beta V} = \ln T + k //$$

C $\frac{P dV}{n_0 dT} = aP$

$$\int dV = a n_0 \int dT$$

$$V = a n_0 T + k //$$

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2.56. An ideal gas has an adiabatic exponent γ . In some process its molar heat capacity varies as $C = \alpha/T$, where α is a constant.

Find:

- (a) the work performed by one mole of the gas during its heating from the temperature T_0 to the temperature η times higher;
 (b) the equation of the process in the variables p, V .

$$n_0 C dT = n_0 C_V dT + p dV$$

$$A \quad C = C_V + \frac{p dV}{n_0 dT}$$

$$B \quad \frac{\alpha}{T} = C_V + \frac{RT}{V} \frac{dV}{dT}$$

$$\int \frac{1}{T} \left(\frac{\alpha}{T} - C_V \right) dT = \int R \frac{dV}{V}$$

$$\Rightarrow \frac{\alpha}{T} = C_V + \frac{p dV}{n_0 dT}$$

$$-\frac{\alpha}{T} - C_V \ln T = R \ln V + k$$

constant another constant
 $k = R \ln k'$

$$\Rightarrow \int p dV = \int \left(\frac{\alpha}{T} - C_V \right) dT$$

$$\Rightarrow -\frac{\alpha R}{PV} - \frac{R}{\gamma-1} \ln \frac{PV}{R} = R \ln k' V$$

$$\Rightarrow W = \alpha \ln \eta - \frac{R T_0 (\eta-1)}{\gamma-1}$$

$$\Rightarrow \ln e^{-\alpha/PV} + \ln \left(\frac{PV}{R} \right)^{\frac{1}{1-\gamma}} = \ln(k' V) \Rightarrow \frac{\alpha(\gamma-1)}{e^{PV}} PV = \text{const}$$

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2.62. Modern vacuum pumps permit the pressures down to $p = 4 \cdot 10^{-15}$ atm to be reached at room temperatures. Assuming that the gas exhausted is nitrogen, find the number of its molecules per 1 cm^3 and the mean distance between them at this pressure.

$$A \quad PV = N k T$$

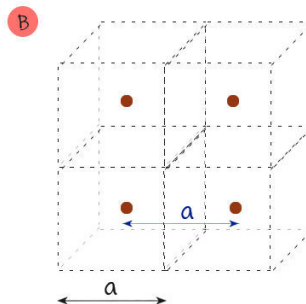
$\therefore$ No. of molecules per unit volume

$$\Rightarrow \frac{N}{V} = \frac{P}{kT}$$

$$= \frac{4 \times 10^{-15} \cdot 10^5}{(1.38 \times 10^{-23}) 300}$$

$$= 10^{11} \text{ molecules/m}^3$$

$$= 10^5 \text{ molecules/cm}^3$$



Lets assume all molecules occupy a cubic shaped volume.

Then distance between them is a

$$V_{av} = a^3$$

$$\frac{1 \text{ m}^3}{10^{11} \text{ molecules}} = a^3$$

$$\Rightarrow a = 10^{-11/3} = 2.2 \times 10^{-4} \text{ m} = 0.22 \text{ mm}$$

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2.63. A vessel of volume $V = 5.0 \text{ l}$ contains $m = 1.4 \text{ g}$ of nitrogen at a temperature $T = 1800 \text{ K}$. Find the gas pressure, taking into account that $\eta = 30\%$ of molecules are disassociated into atoms at this temperature.

$$P = P_{N_2} + P_N \quad (\text{Dalton's law of partial pressures})$$

$$= n_{N_2} \frac{RT}{V} + n_N \frac{RT}{V}$$

$$= \left(\frac{m(1-\eta)}{M_{N_2}} + \frac{m\eta}{\frac{M_{N_2}}{2}} \right) \frac{RT}{V}$$

$$= \frac{m(1+\eta)}{M_{N_2}} \cdot \frac{RT}{V} = \frac{1.4 \times 10^{-3} (1+0.3) (8.314) 1800}{28 \times 10^{-3} \cdot 5 \times 10^{-3}} = 1.9 \times 10^5 \text{ Pa} \\ = 1.9 \text{ atm}$$

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2.64. Under standard conditions the density of the helium and nitrogen mixture equals $\rho = 0.60 \text{ g/l}$. Find the concentration of helium atoms in the given mixture.

For 1 litre mixture at STP,

Gas law,

$$PV = nRT$$

He

N₂

x

$(0.6 - x)$

$$10^5 \cdot 10^{-3} = \left(\frac{x}{4} + \frac{0.6-x}{28} \right) 8.314 \cdot 273$$

$$\Rightarrow x = 0.1$$

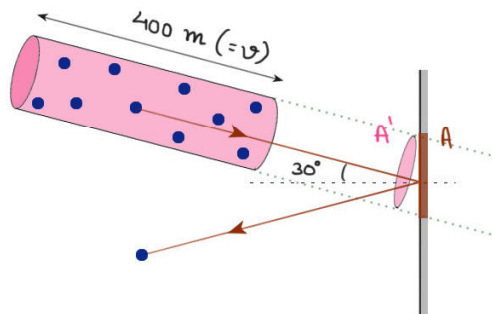
$$\therefore \text{He concentration} = \left(\frac{0.1}{4} \cdot N_A \right) \frac{1}{10^3} \frac{\text{molecules}}{\text{cm}^3} \\ = 1.6 \times 10^{19}$$

Number of molecules per unit volume

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2.65. A parallel beam of nitrogen molecules moving with velocity $v = 400 \text{ m/s}$ impinges on a wall at an angle $\theta = 30^\circ$ to its normal. The concentration of molecules in the beam $n = 0.9 \cdot 10^{19} \text{ cm}^{-3}$. Find the pressure exerted by the beam on the wall assuming the molecules to scatter in accordance with the perfectly elastic collision law.

$F = \text{rate of change of momentum} = \frac{dp}{dt}$



In 1 second, $v \text{ m}$ of beam will strike the surface.

$$\Delta p \text{ one molecule} = 2mv \cos \theta$$

$$\Delta p \text{ in one second for beam} = N \cdot \Delta p$$

$$n(v \cdot A') \cdot 2mv \cos \theta$$

$$\text{Pressure, } P = \frac{F}{A} = \frac{\left(\frac{dp}{dt}\right)}{A} = \frac{nvA'}{A} \cdot 2mv \cos \theta$$

$$= 2nmv^2 \cos^2 \theta$$

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2.66. How many degrees of freedom have the gas molecules, if under standard conditions the gas density is $\rho = 1.3 \text{ mg/cm}^3$ and the velocity of sound propagation in it is $v = 330 \text{ m/s}$.

$$v_{\text{sound}} = \sqrt{\frac{\gamma P}{\rho}}$$

$$\text{Sol: } \gamma = \frac{3v_s^2}{P} = \frac{1.3 \times 10^{-6}}{10^{-6}} \cdot \frac{330^2}{10^5} = 1.4$$

$$C_v = \frac{fR}{2} = \frac{R}{\gamma-1} \Rightarrow f = \frac{2}{\gamma-1}$$

$$\therefore f = \frac{2}{\gamma-1} = \frac{2}{0.4} = 5 //$$

2.67. Determine the ratio of the sonic velocity v in a gas to the root mean square velocity of molecules of this gas, if the molecules are

$$\gamma = 1 + \frac{2}{f}$$

(a) monatomic; (b) rigid diatomic. $\rightarrow$ No vibrations, so $\text{dof} = 5$

$$v_{\text{sound}}^2 = \frac{\gamma P}{\rho} = \frac{\gamma}{\frac{m}{V}} \cdot \frac{nRT}{V} = \frac{\gamma}{\frac{m}{M}} \cdot \frac{RT}{V} = \frac{\gamma RT}{M}$$

$$\text{Required ratio} = \frac{v_{\text{sound}}}{v_{\text{rms}}} = \frac{\sqrt{\frac{\gamma RT}{M}}}{\sqrt{\frac{3RT}{M}}} = \sqrt{\frac{\gamma}{3}}$$

A monatomic, $\gamma = 1 + \frac{2}{f} = 1 + \frac{2}{3} = \frac{5}{3} \Rightarrow \text{Ratio} = \sqrt{\frac{5}{9}}$

B Rigid diatomic, $\gamma = 1 + \frac{2}{f} = 1 + \frac{2}{5} = \frac{7}{5} \Rightarrow \text{Ratio} = \sqrt{\frac{7}{15}}$

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Degrees of freedom

A N atom linear molecule

$$f = \underbrace{3}_{\text{Translational}} + \underbrace{2}_{\text{Rotational}} + \underbrace{2(3N-5)}_{\text{Vibrational}} = 6N-5$$

B N atom non-linear molecule

$$f = \underbrace{3}_{\text{Translational}} + \underbrace{3}_{\text{Rotational}} + \underbrace{2(3N-6)}_{\text{Vibrational}} = 6N-6$$

Total Mean Kinetic Energy of one molecule: $\frac{f}{2} kT$

2.68. A gas consisting of N -atomic molecules has the temperature T at which all degrees of freedom (translational, rotational, and vibrational) are excited. Find the mean energy of molecules in such a gas. What fraction of this energy corresponds to that of translational motion?

Linear: $f = 6N-5$
 $\therefore KE = \frac{6N-5}{2} kT$
 Non-linear: $6N-6$
 $\therefore KE = \frac{6N-6}{2} kT$

$$\frac{\frac{3}{2} kT}{\text{Total KE}} = \begin{cases} \text{Linear: } \frac{3}{6N-5} \\ \text{Non-linear: } \frac{3}{6N-6} \end{cases}$$

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2.69. Suppose a gas is heated up to a temperature at which all degrees of freedom (translational, rotational, and vibrational) of its molecules are excited. Find the molar heat capacity of such a gas in the isochoric process, as well as the adiabatic exponent γ , if the gas consists of

- (a) diatomic; $f = 6N - 5 = 7$
 (b) linear N -atomic; $f = 6N - 5$
 (c) network N -atomic $f = 6N - 6$
- 1 $C_v = \frac{fR}{2}$
 2 $\gamma = 1 + \frac{2}{f}$

2.70. An ideal gas consisting of N -atomic molecules is expanded isobarically. Assuming that all degrees of freedom (translational, rotational, and vibrational) of the molecules are excited, find what fraction of heat transferred to the gas in this process is spent to perform the work of expansion. How high is this fraction in the case of a monatomic gas?

1 Fraction = $\frac{\Delta W}{\Delta Q} = \frac{\Delta Q - \Delta U}{\Delta Q} = \frac{nC_p \Delta T - nC_v \Delta T}{nC_p \Delta T} = \frac{C_p - C_v}{C_p} = 1 - \frac{1}{\gamma} = \frac{2}{f+2}$

2 Fraction(monoatomic) = $\frac{2}{(3)+2} = \frac{2}{5}$

Linear: $\frac{2}{(6N-5)+2} = \frac{2}{6N-3}$
 Non-linear: $\frac{2}{(6N-6)+2} = \frac{1}{3N-2}$

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2.71. Find the molar mass and the number of degrees of freedom of molecules in a gas if its heat capacities are known: $c_v = 0.65 \text{ J/(g} \cdot \text{K)}$ and $c_p = 0.91 \text{ J/(g} \cdot \text{K)}$.

$C_p - C_v = R$
 $C_v = \frac{fR}{2}$

$C_p - C_v = R$
 Should be in $\text{J/mol} \cdot \text{K}$

$\Rightarrow M(C_p - C_v) = R$
 $\Rightarrow M = \frac{R}{C_p - C_v}$
 $= \frac{8.314}{0.91 - 0.65} = 32 \text{ gm/mol}$

$\therefore 1 \text{ gm} \rightarrow C_p \text{ J/g} \cdot \text{K}$
 $\therefore M \text{ gm} \rightarrow MC_p \text{ J/Mg} \cdot \text{K} = C_p \text{ J/mol} \cdot \text{K}$

So, $C_p = MC_p$
 $C_v = MC_v$

Also, $f = \frac{2C_v}{R} = \frac{2MC_v}{R} = \frac{2(32)(0.65)}{8.3} = 5$

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2.72. Find the number of degrees of freedom of molecules in a gas whose molar heat capacity

(a) at constant pressure is equal to $C_p = 29 \text{ J/(mol}\cdot\text{K)}$;

(b) is equal to $C = 29 \text{ J/(mol}\cdot\text{K)}$ in the process $pT = \text{const.}$

$$C_p - C_v = R$$

$$C_v = \frac{fR}{2}$$

A $f = \frac{2C_v}{R} \rightarrow C_p - R$

$$\Rightarrow f = 2\left(\frac{C_p}{R} - 1\right) //$$

B $pT = \text{const}$

$$\Rightarrow p(pV) = \text{const}$$

$$\Rightarrow pV^{1/2} = \text{const} \rightarrow \text{A polytropic process}$$

We know, for a polytropic process

$$C = R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right]$$

$\downarrow \quad \downarrow$
 $1 + \frac{2}{5} \quad \frac{1}{2}$

$$\Rightarrow C = R \left[\frac{f}{2} - \frac{1}{\frac{1}{2}-1} \right]$$

$$\Rightarrow f = 2\left(\frac{C}{R} - 2\right) = 2\left(\frac{29}{8.3} - 2\right) = 3 //$$

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2.73. Find the adiabatic exponent γ for a mixture consisting of n_1 moles of a monatomic gas and n_2 moles of gas of rigid diatomic molecules.

No vibrations

For a monatomic gas, $C_{v1} = \frac{fR}{2} = \frac{3R}{2}$

For a diatomic gas, $C_{v2} = \frac{fR}{2} = \frac{5R}{2}$

$$C_{v\text{mix}} = \frac{n_1 C_{v1} + n_2 C_{v2}}{n_1 + n_2} = \frac{(3n_1 + 5n_2)R}{2(n_1 + n_2)}$$

$$\gamma_{\text{mix}} = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} = 1 + \frac{R}{C_v} = 1 + \frac{2(n_1 + n_2)}{3n_1 + 5n_2} = \frac{5n_1 + 7n_2}{3n_1 + 5n_2} //$$

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2.74. A thermally insulated vessel with gaseous nitrogen at a temperature $t = 27^\circ\text{C}$ moves with velocity $v = 100\text{ m/s}$. How much (in per cent) and in what way will the gas pressure change on a sudden stoppage of the vessel?

As vessel is insulated, $Q = 0$.

As V is constant, $W = 0$.

$\therefore$ Ordered KE gets completely converted to random KE on stoppage.

$$\begin{aligned}\frac{1}{2}mv^2 &= nC_v \Delta T \\ \Rightarrow \Delta T &= \frac{mv^2}{2nC_v} \xrightarrow{nM} \frac{fR}{2} = \frac{5R}{2} \\ &= \frac{\cancel{n}Mv^2}{2\cancel{n} \cdot fR} \\ &= \frac{Mv^2}{5R}\end{aligned}$$

$$\frac{P}{T} = \text{const}$$

$$\Rightarrow TdP - PdT = 0$$

$$\Rightarrow \frac{dP}{P} = \frac{dT}{T} = \frac{Mv^2}{5RT} = 2.2\%$$

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2.75. Calculate at the temperature $t = 17^\circ\text{C}$:

(a) the root mean square velocity and the mean kinetic energy of an oxygen molecule in the process of translational motion;

(b) the root mean square velocity of a water droplet of diameter $d = 0.10\text{ }\mu\text{m}$ suspended in the air.

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

$$\text{A } v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.3)(290)}{32 \times 10^{-3}}} = 470\text{ m/s}$$

KE associated with each degree of freedom = $\frac{1}{2}kT$

$$KE = 3\left(\frac{1}{2}kT\right)$$

B A very small suspended water droplet is like a molecule.

$$\therefore v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3kT}{8V}} = \sqrt{\frac{3(1.38 \times 10^{-23})290}{1000 \cdot \frac{4}{3}\pi\left(\frac{0.1}{2} \times 10^{-6}\right)^3}} = 0.15\text{ m/s}$$

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2.76. A gas consisting of rigid diatomic molecules is expanded adiabatically. How many times has the gas to be expanded to reduce the root mean square velocity of the molecules $\eta = 1.50$ times?

$$\begin{aligned}
 v_{\text{rms}} &\propto \sqrt{T} \\
 \Rightarrow \frac{v_1}{v_2} &= \sqrt{\frac{T_1}{T_2}} \\
 \Rightarrow T_2 &= \frac{T_1}{\eta^2}
 \end{aligned}$$

For adiabatic expansion:

$$\begin{aligned}
 pV^\gamma &= \text{const} \\
 \Rightarrow TV^{\gamma-1} &= \text{const} \\
 \Rightarrow T_1 V_1^{\gamma-1} &= T_2 V_2^{\gamma-1} \\
 \Rightarrow \frac{v_2}{v_1} &= \eta^{\frac{2}{\gamma-1}} = \eta^{\frac{2}{(\frac{5}{2})-1}} = \eta^{\frac{4}{3}}
 \end{aligned}$$

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2.77. The mass $m = 15$ g of nitrogen is enclosed in a vessel at a temperature $T = 300$ K. What amount of heat has to be transferred to the gas to increase the root mean square velocity of its molecules $\eta = 2.0$ times?

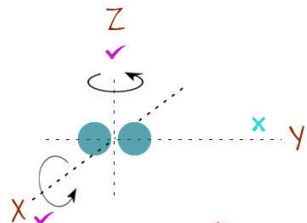
$$\begin{aligned}
 v &\propto \sqrt{T} \\
 \Rightarrow \frac{v_1}{v_2} &= \sqrt{\frac{T_1}{T_2}} \\
 \Rightarrow T_2 &= \eta^2 T_1
 \end{aligned}$$

For an isochoric process:

$$\begin{aligned}
 Q &= n c_v \Delta T \\
 &= \frac{m}{M} \cdot \frac{fR}{2} (T_2 - T_1) \\
 &= \frac{m f R}{2M} (\eta^2 - 1) T_1
 \end{aligned}$$

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2.78. The temperature of a gas consisting of rigid diatomic molecules is $T = 300 \text{ K}$. Calculate the angular root mean square velocity of a rotating molecule if its moment of inertia is equal to $I = 2.1 \cdot 10^{-39} \text{ g} \cdot \text{cm}^2$.



In a diatomic molecule, rotation causes significant energy contribution along two axes only as shown.

- ∴ It has only two rotational degrees of freedom.
- Also, its moment of inertia about both X and Z axis is equal

By equipartition of energy, for each degree of freedom :

$$\frac{1}{2} I \omega_x^2 + \frac{1}{2} I \omega_z^2 = 2 \times \frac{1}{2} kT$$

$$\frac{1}{2} I (\omega_x^2 + \omega_z^2) = kT$$

$$\underbrace{\omega_{\text{rms}}^2}_{\omega_x^2 + \omega_z^2} = \frac{2kT}{I}$$

$$\Rightarrow \omega_{\text{rms}} = \sqrt{\frac{2kT}{I}}$$

Just like, $v_x^2 + v_y^2 + v_z^2 = v_{\text{rms}}^2$

$$\frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2 = \frac{1}{2} m v_{\text{rms}}^2$$

$$= 3 \left(\frac{1}{2} kT \right)$$

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2.79. A gas consisting of rigid diatomic molecules was initially under standard conditions. Then the gas was compressed adiabatically $\eta = 5.0$ times. Find the mean kinetic energy of a rotating molecule in the final state.

$$\text{Mean kinetic energy due to rotation} = 2 \left(\frac{1}{2} kT \right) = k \eta^{\frac{2}{f}} T_0 //$$

For two degrees of freedom of rotation, of a diatomic molecule

$$P V^\gamma = \text{const}$$

$$\Rightarrow T V^{\gamma-1} = \text{const}$$

$$\Rightarrow \frac{T_0}{T_2} = \left(\frac{V_2}{V_0} \right)^{\gamma-1}$$

$$\Rightarrow T_2 = \eta^{\gamma-1} T_0 = \eta^{1+\frac{2}{f}-1} T_0 = \eta^{\frac{2}{f}} T_0$$

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2.83. Calculate the most probable, the mean, and the root mean square velocities of a molecule of a gas whose density under standard atmospheric pressure is equal to $\rho = 1.00 \text{ g/l}$.

$$P = \frac{\rho RT}{M}$$

$$\Rightarrow \frac{RT}{M} = \frac{P}{\rho}$$

$$\therefore v_{mp} = \sqrt{\frac{2P}{\rho}}$$

$$v_{av} = \sqrt{\frac{8P}{\pi\rho}}$$

$$v_{rms} = \sqrt{\frac{3P}{\rho}}$$

$$v_{mp} = \sqrt{\frac{2RT}{M}}$$

$$v_{av} = \sqrt{\frac{8RT}{\pi M}}$$

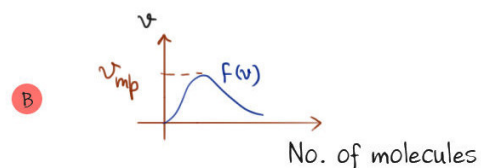
$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

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- 2.85. Determine the gas temperature at which
- the root mean square velocity of hydrogen molecules exceeds their most probable velocity by $\Delta v = 400 \text{ m/s}$;
 - the velocity distribution function $F(v)$ for the oxygen molecules will have the maximum value at the velocity $v = 420 \text{ m/s}$.

$$\text{A} \quad \sqrt{\frac{3RT}{M}} - \sqrt{\frac{2RT}{M}} = \Delta v$$

$$\Rightarrow \sqrt{T} = \Delta v \sqrt{\frac{M}{R}} \frac{1}{\sqrt{3} - \sqrt{2}} //$$



$$\text{Most probable velocity, } v = \sqrt{\frac{2RT}{M}} \Rightarrow T = \frac{M v^2}{2R} //$$

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2.87. At what temperature of a nitrogen and oxygen mixture do the most probable velocities of nitrogen and oxygen molecules differ by $\Delta v = 30 \text{ m/s}$?

$$\sqrt{\frac{2RT}{M_N}} - \sqrt{\frac{2RT}{M_O}} = \Delta v$$

$$\Rightarrow \sqrt{T} = \frac{\Delta v}{\sqrt{2R}} \frac{\sqrt{M_O M_N}}{\sqrt{M_O} - \sqrt{M_N}}$$

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2.92. From the Maxwell distribution function find $\langle v_x^2 \rangle$, the mean value of the squared v_x projection of the molecular velocity in a gas at a temperature T . The mass of each molecule is equal to m .

$$\langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = v_{rms}^2$$

$$\Rightarrow 3 \langle v_x^2 \rangle = v_{rms}^2 \quad \because \text{Random distribution, } v_x = v_y = v_z,$$

$$\Rightarrow \langle v_x^2 \rangle = \frac{v_{rms}^2}{3} = \frac{1}{3} \left(\frac{3kT}{m} \right) = \frac{kT}{m}$$

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2.113. In which case will the efficiency of a Carnot cycle be higher: when the hot body temperature is increased by ΔT , or when the cold body temperature is decreased by the same magnitude?

In carnot cycle is: ☐ ☐ ☐

$$\eta = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}} \\ = 1 - \frac{T_2}{T_1}$$

When hot is made hotter $\eta_h = 1 - \frac{T_2}{T_1 + \Delta T} = \frac{T_1 - T_2 + \Delta T}{T_1 + \Delta T}$

When cold is made colder $\eta_c = 1 - \frac{T_2 - \Delta T}{T_1} = \frac{T_1 - T_2 + \Delta T}{T_1}$

$$\therefore \eta_c > \eta_h //$$

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2.114. Hydrogen is used in a Carnot cycle as a working substance. Find the efficiency of the cycle, if as a result of an adiabatic expansion

☐ ☐ ☐

- (a) the gas volume increases $n = 2.0$ times;
(b) the pressure decreases $n = 2.0$ times.

$$\eta = 1 - \frac{T_2}{T_1}$$

Irrelevant, as efficiency of carnot cycle is independent of working substance.

A $PV^\gamma = \text{const}$

$$\Rightarrow TV^{\gamma-1} = \text{const}$$

$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = \left(\frac{1}{n}\right)^{\gamma-1}$$

$$\therefore \eta = 1 - \frac{1}{n^{\gamma-1}} //$$

B $PV^\gamma = \text{const}$

$$\Rightarrow P^{1-\gamma} T^\gamma = \text{const}$$

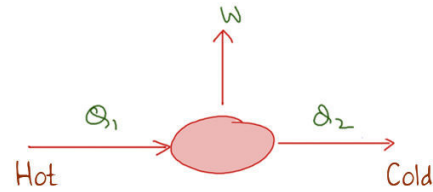
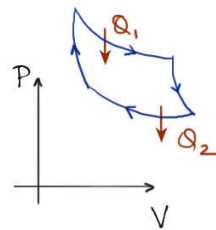
$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma}{\gamma}} = n^{\frac{1-\gamma}{\gamma}}$$

$$\therefore \eta = 1 - n^{\frac{1-\gamma}{\gamma}} //$$

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2.115. A heat engine employing a Carnot cycle with an efficiency of $\eta = 10\%$ is used as a refrigerating machine, the thermal reservoirs being the same. Find its refrigerating efficiency ϵ .

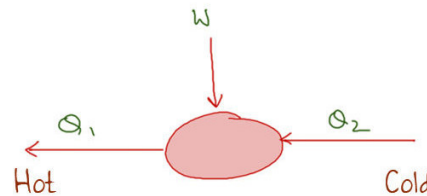
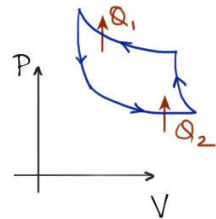
Carnot cycle:



$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

$$\Rightarrow \frac{Q_2}{Q_1} = 1 - \eta$$

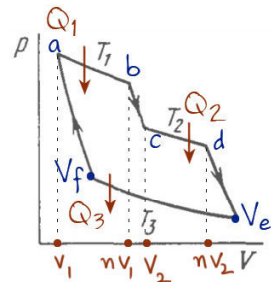
Refrigeration cycle:



$$\epsilon = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{Q_2/Q_1}{1 - Q_2/Q_1} = \frac{1 - \eta}{\eta} //$$

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2.116. An ideal gas goes through a cycle consisting of alternate isothermal and adiabatic curves (Fig. 2.2). The isothermal processes proceed at the temperatures T_1 , T_2 , and T_3 . Find the efficiency of such a cycle, if in each isothermal expansion the gas volume increases in the same proportion.



$$\eta = \frac{W}{Q_{in}} = \frac{Q_1 + Q_2 - |Q_3|}{Q_1 + Q_2} = 1 - \frac{|Q_3|}{Q_1 + Q_2}$$

$$Q_1 = nR T_1 \ln n$$

$$Q_2 = nR T_2 \ln n$$

$$|Q_3| = nR T_3 \ln \frac{V_e}{V_f}$$

$$= 1 - \frac{2 T_3 \ln n}{(T_1 + T_2) \ln n} //$$

For the three adiabatic processes: $T^{\frac{1}{\gamma-1}} V = \text{const}$

$$f-a \quad T_3^{\frac{1}{\gamma-1}} V_f = T_1^{\frac{1}{\gamma-1}} V_1 \quad (1)$$

$$b-c \quad T_1^{\frac{1}{\gamma-1}} nV_1 = T_2^{\frac{1}{\gamma-1}} V_2 \quad (2)$$

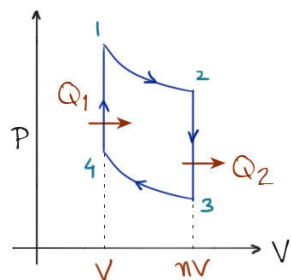
$$d-e \quad T_2^{\frac{1}{\gamma-1}} nV_2 = T_3^{\frac{1}{\gamma-1}} V_e \quad (3)$$

$$\left. \begin{array}{l} (1) \\ (2) \\ (3) \end{array} \right\} \text{Multiply } 1 \times 2 \times 3 \Rightarrow \frac{V_e}{V_f} = n^2$$

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2.117. Find the efficiency of a cycle consisting of two isochoric and two adiabatic lines, if the volume of the ideal gas changes $n = 10$ times within the cycle. The working substance is nitrogen.

$$\begin{aligned} n C_V \Delta T &= C_V (n \Delta T) \\ &= C_V \Delta P V \end{aligned}$$



$$\begin{aligned} \eta &= \frac{W}{Q_1} = \frac{Q_1 - |Q_2|}{Q_1} = 1 - \frac{|Q_2|}{Q_1} = 1 - \frac{C_V (P_2 - P_3) n V}{C_V (P_1 - P_4) V} \\ &= 1 - n^{1-\gamma} \end{aligned}$$

For the two adiabatic processes: $P V^\gamma = \text{const}$

$$\begin{aligned} \Rightarrow 1-2: P_1 V_1^\gamma &= P_2 (nV)^\gamma \quad (1) \\ 3-4: P_4 V_4^\gamma &= P_3 (nV)^\gamma \quad (2) \end{aligned}$$

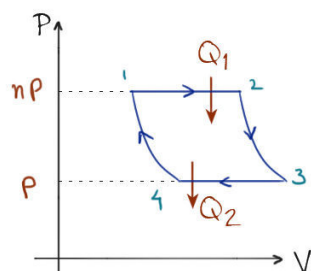
Subtracting 1,2:

$$\begin{aligned} (P_1 - P_4) V_1^\gamma &= (P_2 - P_3) (nV)^\gamma \\ \Rightarrow \frac{P_2 - P_3}{P_1 - P_4} &= \frac{1}{n^\gamma} \end{aligned}$$

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2.118. Find the efficiency of a cycle consisting of two isobaric and two adiabatic lines, if the pressure changes n times within the cycle. The working substance is an ideal gas whose adiabatic exponent is equal to γ .

$$\begin{aligned} \Delta Q &= n C_P \Delta T \\ &= C_P (n \Delta T) \\ &= C_P P \Delta V \end{aligned}$$



$$\begin{aligned} \eta &= \frac{W}{Q_1} = \frac{Q_1 - |Q_2|}{Q_1} = 1 - \frac{|Q_2|}{Q_1} = 1 - \frac{C_P P (V_3 - V_4)}{C_P n P (V_2 - V_1)} \\ &= 1 - n^{\frac{1-\gamma}{\gamma}} \end{aligned}$$

For the two adiabatic process: $P V^\gamma = \text{const}$

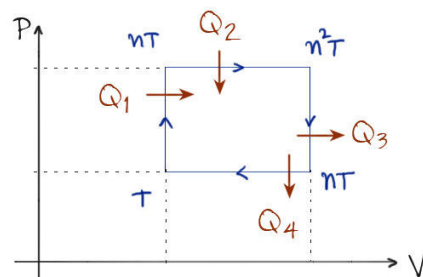
$$\begin{aligned} \Rightarrow 2-3: nP V_2^\gamma &= P V_3^\gamma \Rightarrow n^{1/\gamma} V_2 = V_3 \quad (1) \\ 4-1: nP V_4^\gamma &= P V_1^\gamma \Rightarrow n^{1/\gamma} V_4 = V_1 \quad (2) \end{aligned}$$

Subtracting 1,2:

$$\begin{aligned} (V_2 - V_1) n^{1/\gamma} &= V_3 - V_4 \\ \Rightarrow \frac{V_3 - V_4}{V_2 - V_1} &= n^{1/\gamma} \end{aligned}$$

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2.119. An ideal gas whose adiabatic exponent equals γ goes through a cycle consisting of two isochoric and two isobaric lines. Find the efficiency of such a cycle, if the absolute temperature of the gas rises n times both in the isochoric heating and in the isobaric expansion.



$$\eta = \frac{W}{Q_1 + Q_2} = \frac{Q_1 + Q_2 - |Q_3 + Q_4|}{Q_1 + Q_2} = 1 - \frac{|Q_3 + Q_4|}{Q_1 + Q_2}$$

$$\begin{aligned} Q_1 &= n_0 c_v (n-1) T & |Q_3| &= n_0 c_v (n^2 - n) T \\ Q_2 &= n_0 c_p (n^2 - n) T & |Q_4| &= n_0 c_p (n-1) T \end{aligned}$$

$$\therefore \eta = 1 - \frac{c_v (n^2 - n) + c_p (n-1)}{c_v (n-1) + c_p (n^2 - n)} = 1 - \frac{\cancel{c_v} (n-1) [n + \frac{c_p}{c_v}]}{\cancel{c_v} (n-1) [1 + \frac{c_p}{c_v} n]} = 1 - \frac{n + \gamma}{1 + n\gamma}$$

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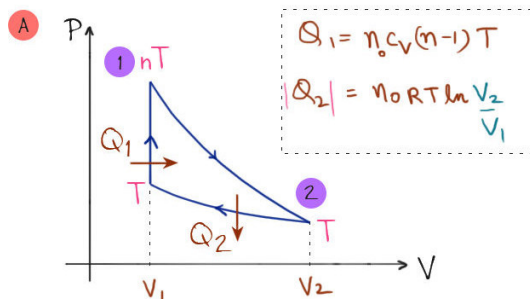
2.120. An ideal gas goes through a cycle consisting of

(a) isochoric, adiabatic, and isothermal lines;

(b) isobaric, adiabatic, and isothermal lines,

with the isothermal process proceeding at the minimum temperature of the whole cycle.

Find the efficiency of each cycle if the absolute temperature varies n -fold within the cycle.



$$Q_1 = n_0 c_v (n-1) T$$

$$|Q_2| = n_0 R T \ln \frac{V_2}{V_1}$$

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - |Q_2|}{Q_1} = 1 - \frac{|Q_2|}{Q_1}$$

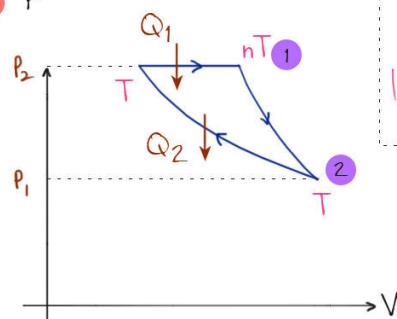
For adiabatic process 1-2:

$$(nT)^{\frac{1}{\gamma-1}} V_1 = T^{\frac{1}{\gamma-1}} V_2 \Rightarrow \frac{V_2}{V_1} = n^{\frac{1}{\gamma-1}}$$

$$\begin{aligned} \therefore \eta &= 1 - \frac{n_0 R T \ln n}{n_0 c_v (n-1) T} \\ &= 1 - \frac{\frac{1}{\gamma-1} \ln n}{\frac{1}{\gamma-1} (n-1)} = 1 - \frac{\ln n}{n-1} \end{aligned}$$

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B



$$Q_1 = n_0 C_p (n-1) T$$

$$|Q_2| = n_0 R T \ln \frac{P_2}{P_1}$$

$$\eta = 1 - \frac{|Q_2|}{Q_1}$$

For adiabatic process 1-2:

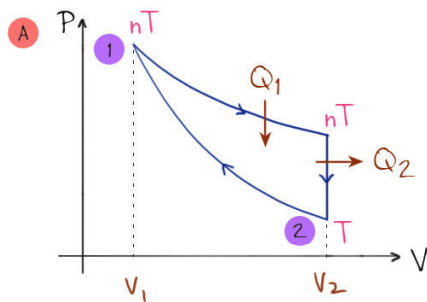
$$P_1 (nT)^{\frac{n}{1-n}} = P_2 T^{\frac{n}{1-n}} \Rightarrow \frac{P_2}{P_1} = n^{\frac{n}{1-n}}$$

$$\therefore \eta = 1 - \frac{n_0 R \frac{n}{1-n} (n-1) T}{n_0 R T \ln n^{\frac{n}{1-n}}}$$

$$= 1 - \frac{\frac{n(n-1)}{1-n}}{\frac{n}{1-n} \ln n} = 1 - \frac{\ln n}{n-1}$$

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2.121. The conditions are the same as in the foregoing problem with the exception that the isothermal process proceeds at the maximum temperature of the whole cycle.



$$Q_1 = n_0 R (nT) \ln \frac{V_2}{V_1}$$

$$|Q_2| = n_0 C_v (n-1) T$$

$$\eta = 1 - \frac{|Q_2|}{Q_1}$$

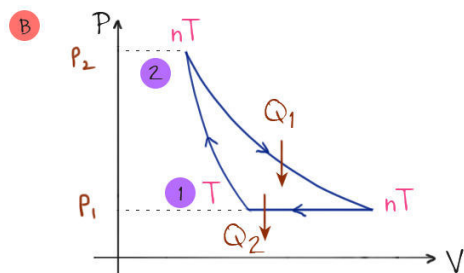
For adiabatic process 2-1:

$$(nT)^{\frac{1}{1-n}} V_1 = T^{\frac{1}{1-n}} V_2 \Rightarrow \frac{V_2}{V_1} = n^{\frac{1}{1-n}}$$

$$\therefore \eta = 1 - \frac{n_0 R \frac{1}{1-n} (n-1) T}{n_0 R (nT) \ln n^{\frac{1}{1-n}}}$$

$$= 1 - \frac{n-1}{n \ln n}$$

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$$Q_1 = n_0 R (nT) \ln \frac{P_2}{P_1}$$

$$|Q_2| = n_0 C_p (n-1) T$$

$$\eta = 1 - \frac{|Q_2|}{Q_1}$$

For adiabatic process 2-1:

$$P_2 (nT)^{\frac{\gamma}{\gamma-1}} = P_1 T^{\frac{\gamma}{\gamma-1}} \Rightarrow \frac{P_2}{P_1} = n^{\frac{\gamma}{\gamma-1}}$$

$$\therefore \eta = 1 - \frac{n_0 R (nT)^{\frac{\gamma}{\gamma-1}} (n-1) T}{n_0 R (nT)^{\frac{\gamma}{\gamma-1}} \ln n^{\frac{\gamma}{\gamma-1}}}$$

$$= 1 - \frac{n-1}{n \ln n}$$

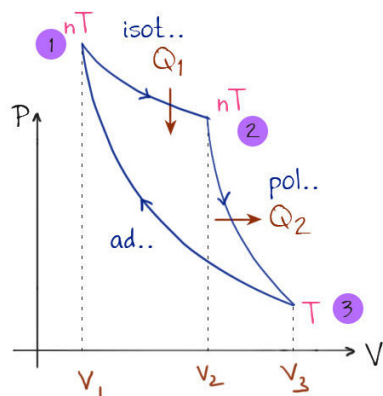
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2.122. An ideal gas goes through a cycle consisting of isothermal, polytropic, and adiabatic lines, with the isothermal process proceeding at the maximum temperature of the whole cycle. Find the efficiency of such a cycle if the absolute temperature varies n -fold within the cycle.

For: $PV^\alpha = \text{const}$

Case 1 $\alpha > \gamma$

$\therefore$ polytropic curve is sharper



$$\eta = 1 - \frac{|Q_2|}{Q_1}$$

$$|Q_2| = n_0 C (n-1) T$$

$$= n_0 R \left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right) (n-1) T$$

$$Q_1 = n_0 R (nT) \ln \frac{V_2}{V_1}$$

For 2-3: $V_2 (nT)^{\frac{1}{\alpha-1}} = V_3 T^{\frac{1}{\alpha-1}}$

For 1-3: $V_1 (nT)^{\frac{1}{\gamma-1}} = V_3 T^{\frac{1}{\gamma-1}}$

dividing $\frac{V_2}{V_1} = n^{\left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right)}$

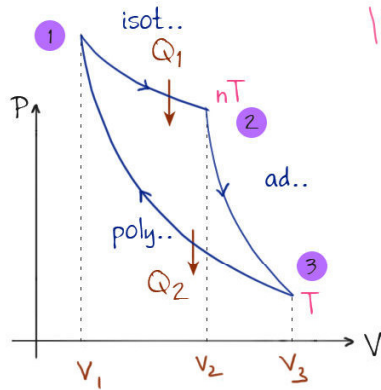
$$\therefore \eta = 1 - \frac{n_0 R \left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right) (n-1) T}{n_0 R nT \left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right) \ln n} = 1 - \frac{n-1}{n \ln n}$$

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Case 2 $\gamma > \alpha$

The previous equations will change to..

∴ adiabatic curve is sharper



$$|Q_2| = n_0 R \left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1} \right) (n-1) T \Rightarrow |Q_2| = n_0 R \left(\frac{1}{\alpha-1} - \frac{1}{\gamma-1} \right) (n-1) T$$

∴ $|Q_2|$ is +ve and $\gamma > \alpha$

$$\text{For 2-3: } V_2 (nT)^{\frac{1}{\alpha-1}} = V_3 T^{\frac{1}{\alpha-1}} \Rightarrow V_2 (nT)^{\frac{1}{\gamma-1}} = V_3 T^{\frac{1}{\gamma-1}}$$

$$\text{For 1-3: } V_1 (nT)^{\frac{1}{\gamma-1}} = V_3 T^{\frac{1}{\gamma-1}} \Rightarrow V_1 (nT)^{\frac{1}{\alpha-1}} = V_3 T^{\frac{1}{\alpha-1}}$$

$$\Rightarrow \frac{V_2}{V_1} = n^{\frac{1}{\alpha-1} - \frac{1}{\gamma-1}}$$

$$\therefore \eta = 1 - \frac{|Q_2|}{Q_1} = 1 - \frac{n_0 R \left(\frac{1}{\alpha-1} - \frac{1}{\gamma-1} \right) (n-1) T}{n_0 R n T \left(\frac{1}{\alpha-1} - \frac{1}{\gamma-1} \right) \ln n} = 1 - \frac{n-1}{n \ln n}$$

Its still $n_0 R (nT) \ln \frac{V_2}{V_1}$

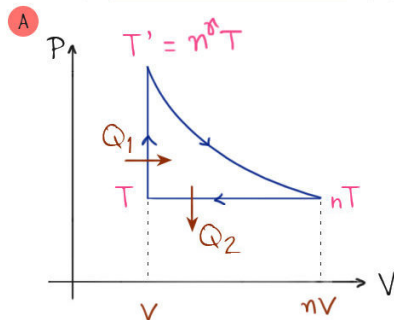
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2.123. An ideal gas with the adiabatic exponent γ goes through a direct (clockwise) cycle consisting of **adiabatic, isobaric, and isochoric** lines. Find the efficiency of the cycle if in the adiabatic process the **volume** of the ideal gas

(a) increases n -fold; (b) decreases n -fold.

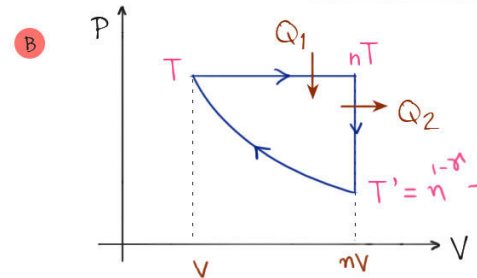
In adiabatic process,

$$T V^{\gamma-1} = \text{const}$$



$$\eta = 1 - \frac{|Q_2|}{Q_1} = 1 - \frac{n_0 C_p (n-1) T}{n_0 C_v (n^\gamma - 1) T} = 1 - \frac{\gamma (n-1)}{n^\gamma - 1}$$

$$T' V^{\gamma-1} = (nT) (nV)^{\gamma-1} \Rightarrow T' = n^\gamma T$$



$$T' (nV)^{\gamma-1} = T V^{\gamma-1} \Rightarrow T' = n^{1-\gamma} T$$

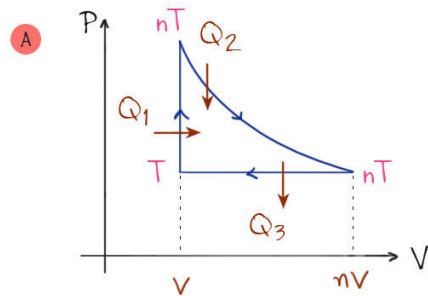
$$\eta = 1 - \frac{|Q_2|}{Q_1} = 1 - \frac{n_0 C_v (n - n^{1-\gamma}) T}{n_0 C_p (n-1) T}$$

$$= 1 - \frac{n (1 - n^{-\gamma})}{\gamma (n-1)}$$

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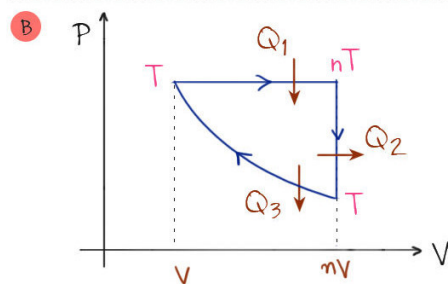
2.124. Calculate the efficiency of a cycle consisting of isothermal, isobaric, and isochoric lines, if in the isothermal process the volume of the ideal gas with the adiabatic exponent γ

(a) increases n -fold; (b) decreases n -fold.



$$\eta = \frac{W}{Q_1 + Q_2} = \frac{Q_1 + Q_2 - |Q_3|}{Q_1 + Q_2} = 1 - \frac{|Q_3|}{Q_1 + Q_2}$$

$$= 1 - \frac{n_0 C_V (n-1) T}{n_0 C_V (n-1) T + n_0 R (nT) \ln \frac{nV}{V}} = 1 - \frac{\gamma (n-1)}{n-1 + (\gamma-1)n \ln n} //$$

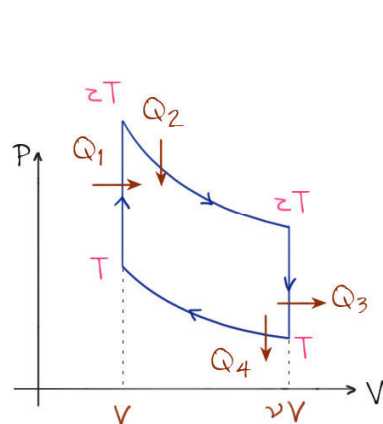


$$\eta = \frac{W}{Q_1} = \frac{Q_1 - |Q_2| - |Q_3|}{Q_1} = 1 - \frac{(|Q_2| + |Q_3|)}{Q_1}$$

$$= 1 - \frac{n_0 C_V (n-1) T + n_0 R T \ln \frac{nV}{V}}{n_0 C_P (n-1) T} = 1 - \frac{n-1 + (\gamma-1) \ln n}{\gamma (n-1)} //$$

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2.125. Find the efficiency of a cycle consisting of two isochoric and two isothermal lines if the volume varies v -fold and the absolute temperature τ -fold within the cycle. The working substance is an ideal gas with the adiabatic exponent γ .



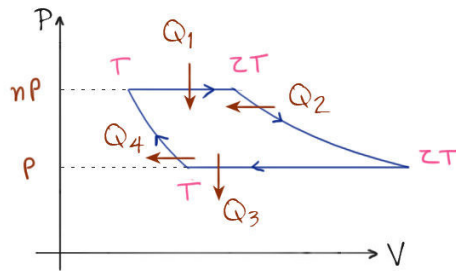
$$\eta = 1 - \frac{|Q_3| + |Q_4|}{Q_1 + Q_2}$$

$$= 1 - \frac{n_0 C_V (z-1) T + n_0 R T \ln v}{n_0 C_V (z-1) T + n_0 R (zT) \ln v}$$

$$= 1 - \frac{z-1 + (\gamma-1) \ln v}{z-1 + (\gamma-1) z \ln v} //$$

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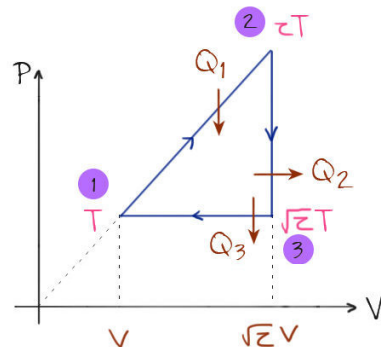
2.126. Find the efficiency of a cycle consisting of two isobaric and two isothermal lines if the pressure varies n -fold and the absolute temperature τ -fold within the cycle. The working substance is an ideal gas with the adiabatic exponent γ .



$$\begin{aligned}\eta &= 1 - \frac{|Q_3| + |Q_4|}{Q_1 + Q_2} \\ &= 1 - \frac{n_0 c_p (\tau - 1) T + n_0 R T \ln n}{n_0 c_p (\tau - 1) T + n_0 R (\tau T) \ln n} \\ &= 1 - \frac{\gamma(\tau - 1) + (\gamma - 1) \ln n}{\gamma(\tau - 1) + \tau(\gamma - 1) \ln n}\end{aligned}$$

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2.127. An ideal gas with the adiabatic exponent γ goes through a cycle (Fig. 2.3) within which the absolute temperature varies τ -fold. Find the efficiency of this cycle.



During 1-2:

$$\begin{aligned}P &\propto V \\ \Rightarrow P V^{-1} &= \text{const} \\ \Rightarrow T V^{-2} &= \text{const} \\ \Rightarrow T V_1^{-2} &= \tau T V_2^{-2} \\ \Rightarrow V_2 &= \sqrt{\tau} V_1\end{aligned}$$

$$\therefore T_3 = \sqrt{\tau} T_1$$

$$\left(\text{As } \frac{V_2}{V_1} = \frac{T_3}{T_1} \right)$$

$$\begin{aligned}\eta &= 1 - \frac{|Q_2| + |Q_3|}{Q_1} \\ &= 1 - \frac{n_0 c_v (\tau - \sqrt{\tau}) T + n_0 c_p (\sqrt{\tau} - 1) T}{n_0 R \left[\frac{1}{\gamma - 1} - \frac{1}{(-1) - 1} \right] (\tau - 1) T} \\ &= 1 - \frac{(\sqrt{\tau} - 1) [\sqrt{\tau} + \gamma]}{(\tau - 1) (\gamma + 1) \sqrt{\tau} + 1} \\ &= 1 - \frac{(\sqrt{\tau} + \gamma)^2}{(\sqrt{\tau} + 1) (\gamma + 1)}\end{aligned}$$

Youtube / Arihant Senior

2.130. Find the entropy increment of one mole of carbon dioxide when its absolute temperature increases $n = 2.0$ times if the process of heating is

(a) isochoric; (b) isobaric.

The gas is to be regarded as ideal.

A Entropy change in Isochoric process

$$\begin{aligned} dS &= \frac{dQ}{T} \\ &= \frac{n_0 c_v dT}{T} \\ \Delta S &= \int dS = n_0 c_v \int_{T_0}^{nT_0} \frac{dT}{T} \\ &= n_0 c_v \ln n \end{aligned}$$

B Entropy change in Isobaric process

$$\begin{aligned} dS &= \frac{dQ}{T} \\ &= \frac{n_0 c_p dT}{T} \\ \Delta S &= \int dS = n_0 c_p \int_{T_0}^{nT_0} \frac{dT}{T} \\ &= n_0 c_p \ln n \end{aligned}$$

Youtube / Arihant Senior

2.131. The entropy of $v = 4.0$ moles of an ideal gas increases by $\Delta S = 23 \text{ J/K}$ due to the isothermal expansion. How many times should the volume $v = 4.0$ moles of the gas be increased?

Entropy change in Isothermal process

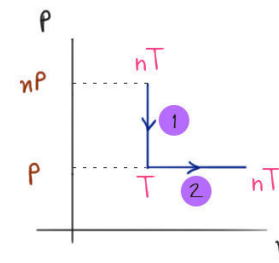
$$\begin{aligned} \Delta S &= \frac{\Delta Q}{T_0} \\ &= \frac{n_0 R \cancel{T_0} \ln \frac{V_2}{V_1}}{\cancel{T_0}} \\ &= n_0 R \ln \frac{V_2}{V_1} = n_0 R \ln \frac{P_1}{P_2} \end{aligned}$$

$$\Rightarrow \frac{V_2}{V_1} = e^{\frac{\Delta S}{n_0 R}}$$

Youtube / Arihant Senior

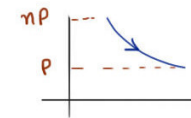
2.132. Two moles of an ideal gas are cooled isochorically and then expanded isobarically to lower the gas temperature back to the initial value. Find the entropy increment of the gas if in this process the gas pressure changed $n = 3.3$ times.

$$\begin{aligned}\Delta S &= \Delta S_1 + \Delta S_2 \\ &= n_0 c_v \ln \frac{T}{nT} + n_0 c_p \ln \frac{nT}{T} \quad (\text{Derived in 2.130}) \\ &= \frac{n_0 R}{\cancel{\gamma-1}} \ln n (\cancel{\gamma-1}) = n_0 R \ln n //\end{aligned}$$



Method 2:

Lets take Isothermal path between same initial and final states



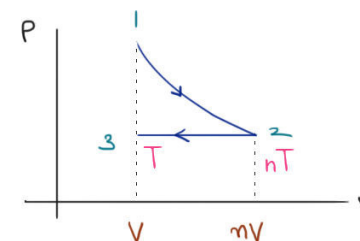
$$\Delta S = n_0 R \ln \frac{nP}{P} = n_0 R \ln n //$$

(derived in 2.131)

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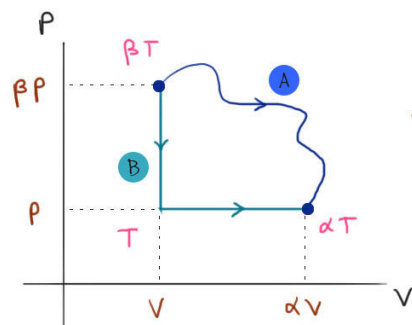
2.133. Helium of mass $m = 1.7$ g is expanded adiabatically $n = 3.0$ times and then compressed isobarically down to the initial volume. Find the entropy increment of the gas in this process.

$$\begin{aligned}\Delta S &= \cancel{\Delta S_{12}}^0 + \Delta S_{23} \\ &= 0 + n_0 c_p \ln \frac{V}{nV} \\ &= -\gamma \frac{m}{M} \frac{R}{\gamma-1} \ln n //\end{aligned}$$



Youtube / Arihant Senior

2.134. Find the entropy increment of $\nu = 2.0$ moles of an ideal gas whose adiabatic exponent $\gamma = 1.30$ if, as a result of a certain process, the gas volume increased $\alpha = 2.0$ times while the pressure dropped $\beta = 3.0$ times.



∴ Entropy is state function,

$$\Delta S_{\text{path A}} = \Delta S_{\text{path B}}$$

$$= n_0 C_V \ln \frac{T}{\beta T} + n_0 C_P \ln \frac{\alpha T}{T} \quad (\text{Derived in 2.130})$$

$$= \frac{n_0 R}{\gamma - 1} (\gamma \ln \alpha - \ln \beta)$$

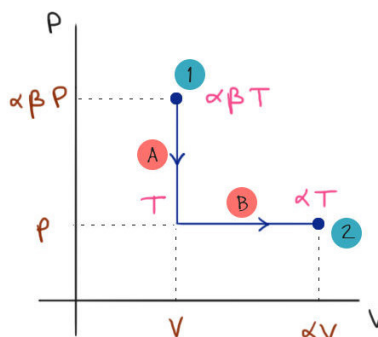
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2.135. Vessels 1 and 2 contain $\nu = 1.2$ moles of gaseous helium. The ratio of the vessels' volumes $V_2/V_1 = \alpha = 2.0$, and the ratio of the absolute temperatures of helium in them $T_1/T_2 = \beta = 1.5$. Assuming the gas to be ideal, find the difference of gas entropies in these vessels, $S_2 - S_1$.

∴ $n_1 = n_2$, let's represent the process on PV curve

$$\frac{V_2}{V_1} \cdot \frac{T_1}{T_2} = \alpha \beta$$

$$\Rightarrow \frac{P_1}{P_2} = \alpha \beta$$



$$S_2 - S_1 = \Delta S_{1-2}$$

$$= \Delta S_A + \Delta S_B$$

$$= n_0 C_V \ln \frac{T}{\alpha \beta T} + n_0 C_P \ln \frac{\alpha T}{T}$$

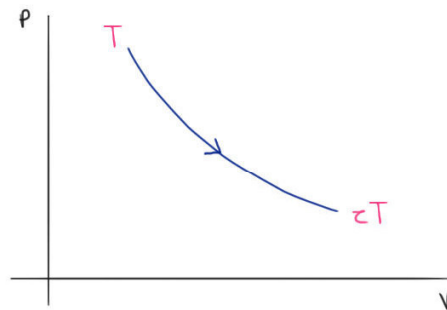
$$= \frac{n_0 R}{\gamma - 1} [-\ln \alpha \beta + \gamma \ln \alpha]$$

$$= \frac{n_0 R}{\gamma - 1} [(\gamma - 1) \ln \alpha - \ln \beta]$$

$$= n_0 R \left[\ln \alpha - \frac{\ln \beta}{\gamma - 1} \right]$$

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2.136. One mole of an ideal gas with the adiabatic exponent γ goes through a polytropic process as a result of which the absolute temperature of the gas increases τ -fold. The polytropic constant equals n . Find the entropy increment of the gas in this process.



For a polytropic process $pV^n = \text{const}$,

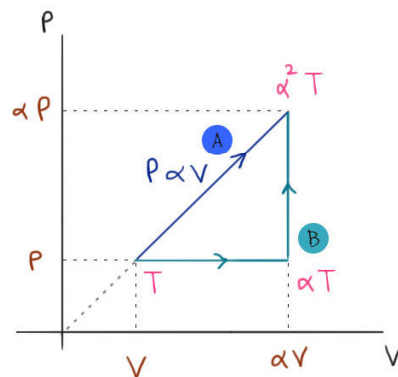
$$dQ = n_0 C dT \quad R \left(\frac{1}{\gamma-1} - \frac{1}{n-1} \right)$$

$$\Rightarrow dS = \frac{dQ}{T} = n_0 R C \frac{dT}{T}$$

$$\Rightarrow \Delta S = \int dS = n_0 R C \int_T^{\tau T} \frac{dT}{T} = n_0 C \ln \tau //$$

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2.137. The expansion process of $\nu = 2.0$ moles of argon proceeds so that the gas pressure increases in direct proportion to its volume. Find the entropy increment of the gas in this process provided its volume increases $\alpha = 2.0$ times. (Similar to 2.134)



∴ Entropy is state function,

$$\Delta S_A = \Delta S_B$$

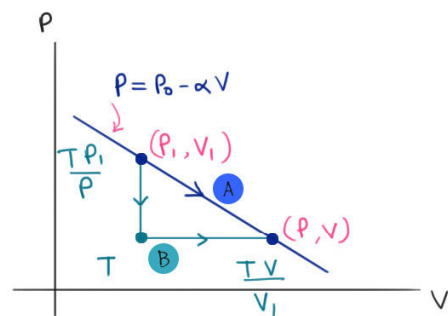
$$= n_0 C_p \ln \frac{\alpha T}{T} + n_0 C_v \ln \frac{\alpha^2 T}{\alpha T}$$

$$= \frac{n_0 R}{\gamma-1} (\gamma \ln \alpha + \ln \alpha)$$

$$= n_0 R \frac{(\gamma+1)}{(\gamma-1)} \ln \alpha //$$

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2.138. An ideal gas with the adiabatic exponent γ goes through a process $p = p_0 - \alpha V$, where p_0 and α are positive constants, and V is the volume. At what volume will the gas entropy have the maximum value?



Lets take a random reference state (P_1, V_1) on the process.

Now we can calculate entropy at a general state (P, V) wrt (P_1, V_1) , and maximize it.

$$S_{P,V} - S_{P_1,V_1} = \Delta S_{\text{path A}} = \Delta S_{\text{path B}}$$

$$= n_0 C_V \ln \frac{T}{T^{1/\gamma} P} + n_0 C_P \ln \frac{T V / V_1}{T}$$

$$= \frac{n_0 R}{\gamma - 1} \left[\ln \frac{P_0 - \alpha V}{P_1} + \gamma \ln \frac{V}{V_1} \right]$$

$$\therefore S_{P,V} = S_{P_1,V_1} + \frac{n_0 R}{\gamma - 1} \left[\ln \frac{P_0 - \alpha V}{P_1} + \gamma \ln \frac{V}{V_1} \right] = f(V)$$

$$S_{P,V \text{ max}} \Rightarrow f'(V) = 0 \Rightarrow -\alpha \frac{1}{P_0 - \alpha V} + \frac{\gamma}{V} = 0 \Rightarrow V = \frac{P_0 \gamma}{\alpha (1 + \gamma)} //$$

Youtube / Arihant Senior

2.139. One mole of an ideal gas goes through a process in which the entropy of the gas changes with temperature T as $S = aT + C_V \ln T$, where a is a positive constant, C_V is the molar heat capacity of this gas at constant volume. Find the volume dependence of the gas temperature in this process if $T = T_0$ at $V = V_0$.

Given,

$$S = aT + C_V \ln T$$

$$\Rightarrow dS = a dT + \frac{C_V}{T} dT$$

$$\therefore dQ = T dS$$

$$= C_V dT + aT dT \quad (1)$$

Also, from our standard energy equation:

$$dQ = C_V dT + P dV \quad (2)$$

Comparing 1 and 2:

$$aT dT = P dV$$

$$\Rightarrow aT dT = \frac{RT}{V} dV$$

$$\Rightarrow a \int_{T_0}^T dT = R \int_{V_0}^V \frac{dV}{V}$$

$$\Rightarrow a(T - T_0) = R \ln \frac{V}{V_0}$$

$$\Rightarrow T = T_0 + \frac{R}{a} \ln \frac{V}{V_0} //$$

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2.142. At very low temperatures the heat capacity of crystals is equal to $C = aT^3$, where a is a constant. Find the entropy of a crystal as a function of temperature in this temperature interval.

$$dS = \frac{dQ}{T} = \frac{C dT}{T} = \frac{aT^3 dT}{T} = aT^2 dT$$

$$S = \int_0^S dS = a \int_0^T T^2 dT = \frac{aT^3}{3} //$$

(As we can take $S=0$ at $T=0$.)

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2.143. Find the entropy increment of an aluminum bar of mass $m = 3.0$ kg on its heating from the temperature $T_1 = 300$ K up to $T_2 = 600$ K if in this temperature interval the specific heat capacity of aluminum varies as $c = a + bT$, where $a = 0.77$ J/(g·K), $b = 0.46$ mJ/(g·K²).

$$dQ = mc dT$$

$$dS = \frac{dQ}{T} = mc \frac{dT}{T}$$

$$\begin{aligned} \Rightarrow \Delta S &= \int_{T_1}^{T_2} m(a+bT) \frac{dT}{T} \\ &= ma \ln \frac{T_2}{T_1} + mb(T_2 - T_1) // \end{aligned}$$

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2.144. In some process the temperature of a substance depends on its entropy S as $T = aS^n$, where a and n are constants. Find the corresponding heat capacity C of the substance as a function of S . At what condition is $C < 0$?

$$S = \left(\frac{T}{a}\right)^{\frac{1}{n}}$$

$$\Rightarrow dS = \frac{1}{n} a^{-\frac{1}{n}} T^{\frac{1}{n}-1} dT$$

We know,

$$dq = CdT = TdS$$

$$\Rightarrow C = \frac{TdS}{dT} = \frac{T^{1/n}}{n a^{1/n}} = \frac{S}{n} \Rightarrow C < 0 \text{ if } n < 0$$

as S is always +ve

∴ When entropy increases by lowering temperature, C is -ve.

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2.145. Find the temperature T as a function of the entropy S of a substance for a polytropic process in which the heat capacity of the substance equals C . The entropy of the substance is known to be equal to S_0 at the temperature T_0 . Draw the approximate plots $T(S)$ for $C > 0$ and $C < 0$.

For, $pV^n = \text{const}$

$$C = R \left[\frac{1}{\gamma-1} - \frac{1}{n-1} \right]$$

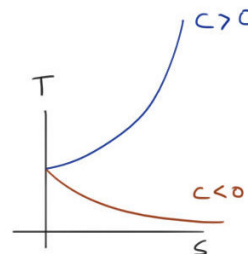
Its constant

$$dS = \frac{dQ}{T} = \frac{C dT}{T}$$

$$\Rightarrow \int_{S_0}^S dS = C \int_{T_0}^T \frac{dT}{T}$$

$$\Rightarrow S - S_0 = C \ln \frac{T}{T_0}$$

$$\Rightarrow T = T_0 e^{\frac{S-S_0}{C}}$$



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2.146. One mole of an ideal gas with heat capacity C_V goes through a process in which its entropy S depends on T as $S = \alpha/T$, where α is a constant. The gas temperature varies from T_1 to T_2 . Find:

- (a) the molar heat capacity of the gas as a function of its temperature;
 (b) the amount of heat transferred to the gas;
 (c) the work performed by the gas.

A $S = \frac{\alpha}{T}$
 $ds = -\frac{\alpha}{T^2} dT = \frac{dQ}{T}$
 $\Rightarrow \frac{C dT}{T} = -\frac{\alpha}{T^2} dT$
 $\Rightarrow C = \frac{-\alpha}{T}$

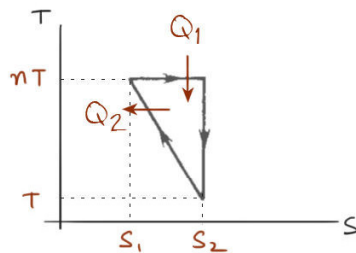
B $dQ = C dT$
 $\int dQ = -\alpha \int_{T_1}^{T_2} \frac{dT}{T^2}$
 $\Rightarrow \Delta Q = -\alpha \ln \frac{T_2}{T_1}$

C $W = \Delta Q - C_V \Delta T$
 $= -\alpha \ln \frac{T_2}{T_1} - C_V (T_2 - T_1)$

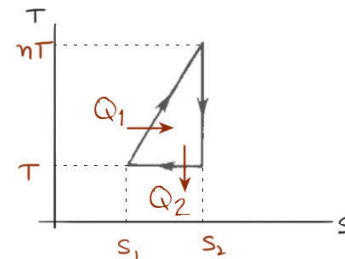
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2.147. A working substance goes through a cycle within which the absolute temperature varies n -fold, and the shape of the cycle is shown in (a) Fig. 2.4a; (b) Fig. 2.4b, where T is the absolute temperature, and S the entropy. Find the efficiency of each cycle.

Q is area under T - S curve.



A $\eta = 1 - \frac{|Q_2|}{Q_1}$
 $= 1 - \frac{\frac{1}{2}(nT+T)(S_2-S_1)}{nT(S_2-S_1)}$
 $= 1 - \frac{n+1}{2n} = \frac{n-1}{2n}$



B $\eta = 1 - \frac{|Q_2|}{Q_1}$
 $= 1 - \frac{T(S_2-S_1)}{\frac{1}{2}(nT+T)(S_2-S_1)}$
 $= 1 - \frac{2}{n+1} = \frac{n-1}{n+1}$

Constant $S \Rightarrow$ zero Q
 Adiabatic

S increases $\Rightarrow$ +ve Q
 S decreases $\Rightarrow$ -ve Q

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2.148. One of the two thermally insulated vessels interconnected by a tube with a valve contains $\nu = 2.2$ moles of an ideal gas. The other vessel is evacuated. The valve having been opened, the gas increased its volume $n = 3.0$ times. Find the entropy increment of the gas.

The entropy change is NOT $dS = \frac{dq}{T}$

The correct equation is $dS = \frac{dq_{rev}}{T}$

Free expansion is irreversible so we can not use the first equation.

However since entropy is a state function, we can calculate change in entropy by taking a reversible route to the same end state.

Note: Heat exchange with surroundings will be different in both cases.

Sol:

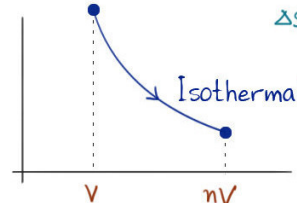
In free expansion:

Work = 0

Q = 0

$\Delta U = 0 \Rightarrow T_i = T_f$

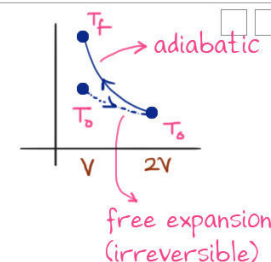
∴ let's take reversible isothermal path



$$\Delta S = \int \frac{dq_{rev}}{T} = \int \frac{p dV}{T} = n_0 R \int \frac{dV}{V} = n_0 R \ln n //$$

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2.149. A weightless piston divides a thermally insulated cylinder into two equal parts. One part contains one mole of an ideal gas with adiabatic exponent γ , the other is evacuated. The initial gas temperature is T_0 . The piston is released and the gas fills the whole volume of the cylinder. Then the piston is slowly displaced back to the initial position. Find the increment of the internal energy and the entropy of the gas resulting from these two processes.



A ΔS

In free expansion: $\Delta S_1 = n_0 R \ln n$ (derived in 2.128)

In adiabatic reversible compression: $\Delta S_2 = 0$ (∵ $dS = \frac{dq_{rev}}{T}$)

∴ $\Delta S = \Delta S_1 + \Delta S_2 = n_0 R \ln 2 //$

B ΔU

In free expansion: $\Delta U_1 = 0$ (∵ $T_i = T_f$ in free expansion)

In adiabatic reversible compression: $\Delta U_2 = n_0 C_V (T_f - T_0) = n_0 C_V T_0 (2^{\gamma-1} - 1)$

∴ $\Delta U = \Delta U_1 + \Delta U_2 = n_0 C_V T_0 (2^{\gamma-1} - 1) //$

$$T_0 (2V)^{\gamma-1} = T_f (V)^{\gamma-1}$$

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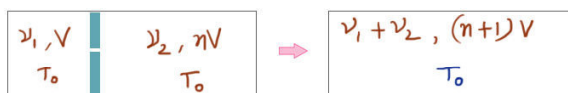
2.150. An ideal gas was expanded from the initial state to the volume V without any heat exchange with the surrounding bodies. Will the final gas pressure be the same in the case of (a) a fast and in the case of (b) a very slow expansion process?

- 1 In free expansion, $Q=0$, $W=0$, $\therefore \Delta U=0 \Rightarrow T_f = T_i$
 - 2 In adiabatic expansion $Q=0$, $W=+ve$, $\therefore \Delta U < 0 \Rightarrow T_f < T_i$
- $P_f V_f = nRT_f \Rightarrow P_{f1} > P_{f2}$
free expansion
adiabatic expansion

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2.151. A thermally insulated vessel is partitioned into two parts so that the volume of one part is $n = 2.0$ times greater than that of the other. The smaller part contains $\nu_1 = 0.30$ mole of nitrogen, and the greater one $\nu_2 = 0.70$ mole of oxygen. The temperature of the gases is the same. A hole is punctured in the partition and the gases are mixed. Find the corresponding increment of the system's entropy, assuming the gases to be ideal.

In free expansion, $Q=0$, $W=0$, $\Delta U=0 \Rightarrow \Delta T=0$



To calculate entropy change, let's take both gases through equivalent reversible isothermal expansion filling whole volume (similar to 2.148).

$$\Delta S = \Delta S_1 + \Delta S_2$$

$$= \frac{\Delta Q_1}{T_0} + \frac{\Delta Q_2}{T_0}$$

$$= \frac{\nu_1 R T_0 \ln \frac{(n+1)V}{V}}{T_0} + \frac{\nu_2 R T_0 \ln \frac{(n+1)V}{nV}}{T_0} = R \left(\nu_1 \ln(n+1) + \nu_2 \ln \frac{n+1}{n} \right)$$

Youtube / Arihant Senior

2.152. A piece of copper of mass $m_1 = 300$ g with initial temperature $t_1 = 97^\circ\text{C}$ is placed into a calorimeter in which the water of mass $m_2 = 100$ g is at a temperature $t_2 = 7^\circ\text{C}$. Find the entropy increment of the system by the moment the temperatures equalize. The heat capacity of the calorimeter itself is negligibly small.

$T_1 = 100$ K, $T_2 = 280$ K. Let the equilibrium temperature be T .

and Specific heat capacities of copper and water be C_1 & C_2

$$\Delta S = \Delta S_1 + \Delta S_2$$

$$= \int \frac{dq_1}{T} + \int \frac{dq_2}{T}$$

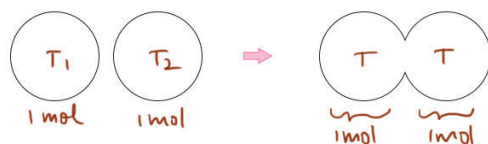
$$= m_1 C_1 \int_{T_1}^T \frac{dT}{T} + m_2 C_2 \int_{T_2}^T \frac{dT}{T}$$

$$= m_1 C_1 \ln \frac{T}{T_1} + m_2 C_2 \ln \frac{T}{T_2} = 3.9 \text{ J/K}$$

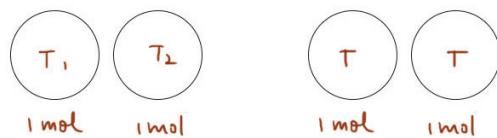
$$\left. \begin{aligned} m_1 C_1 (T_1 - T) &= m_2 C_2 (T - T_2) \\ \Rightarrow T &= \frac{m_1 C_1 T_1 + m_2 C_2 T_2}{m_1 C_1 + m_2 C_2} \end{aligned} \right\} \begin{array}{l} \approx 300 \text{ K} \\ \text{Calorimetry, energy conservation:} \end{array}$$

Youtube / Arihant Senior

2.153. Two identical thermally insulated vessels interconnected by a tube with a valve contain one mole of the same ideal gas each. The gas temperature in one vessel is equal to T_1 and in the other, T_2 . The molar heat capacity of the gas of constant volume equals C_V . The valve having been opened, the gas comes to a new equilibrium state. Find the entropy increment ΔS of the gas. Demonstrate that $\Delta S > 0$.



|||



isochoric

isochoric

As $Q = 0$ and $W = 0$:
 $\Delta U_1 + \Delta U_2 = 0$

$$\Rightarrow 1 C_V (T - T_1) + 1 C_V (T - T_2) = 0$$

$$\Rightarrow T = \frac{T_1 + T_2}{2}$$

$$\therefore \Delta S = \Delta S_1 + \Delta S_2$$

$$= C_V \int_{T_1}^T \frac{dT}{T} + C_V \int_{T_2}^T \frac{dT}{T} = C_V \ln \frac{T^2}{T_1 T_2} = C_V \ln \frac{(T_1 + T_2)^2}{4 T_1 T_2}$$

$$\text{Also, } (T_1 + T_2)^2 > 4 T_1 T_2, \therefore \Delta S > 0 //$$

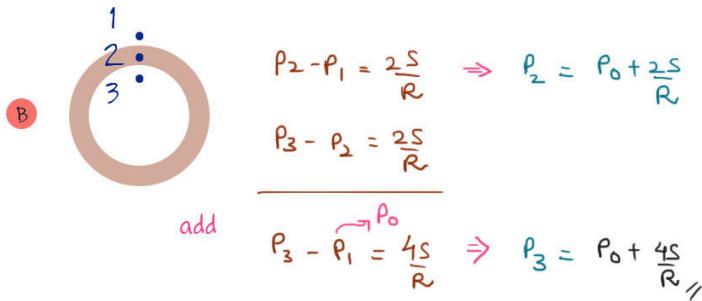
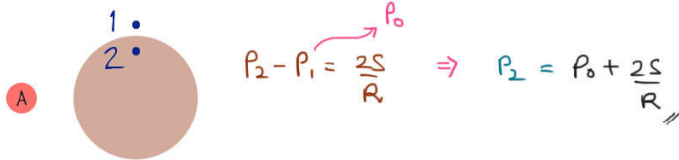
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2.160. Find the pressure

- a) Inside a drop
b) Inside a bubble

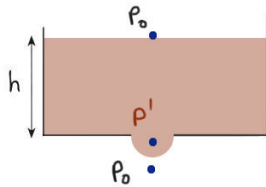
If the surface tension is S and radius R .

We know that pressure difference across a meniscus of radius R is, $\Delta P = \frac{2S}{R}$



Youtube / Arihant Senior

2.161. In the bottom of a vessel with mercury there is a round hole of diameter $d = 70 \mu\text{m}$. At what maximum thickness of the mercury layer will the liquid still not flow out through this hole?



At maximum height, droplet will be of hemispherical shape, as Force due to Surface Tension will be maximum then.

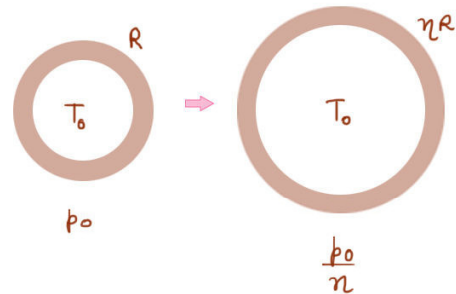
We know that pressure difference across a meniscus of radius R is, $\Delta P = \frac{2S}{R}$

So, $P' = P_0 + \frac{2S}{R}$ & Also from top, $P_0 + \rho gh = P'$

$\therefore P_0 + \frac{2S}{R} = P_0 + \rho gh \Rightarrow h = \frac{2S}{\rho g R}$

Youtube / Arihant Senior

2.162. A vessel filled with air under pressure p_0 contains a soap bubble of diameter d . The air pressure having been reduced isothermally n -fold, the bubble diameter increased η -fold. Find the surface tension of the soap water solution.



Since the process is isothermal. Inside the soap:

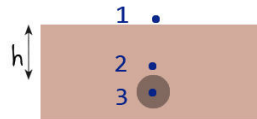
$$p_1 v_1 = p_2 v_2$$

$$\Rightarrow \left(p_0 + \frac{4S}{R} \right) \frac{4}{3} \pi R^3 = \left(\frac{p_0}{n} + \frac{4S}{\eta R} \right) \frac{4}{3} \pi (\eta R)^3$$

$$\Rightarrow S = \frac{R p_0 \left(1 - \frac{\eta^3}{n} \right)}{4(\eta^2 - 1)}$$

Youtube / Arihant Senior

2.163. Find the pressure in an air bubble of diameter $d = 4.0 \mu\text{m}$, located in water at a depth $h = 5.0 \text{ m}$. The atmospheric pressure has the standard value p_0 .



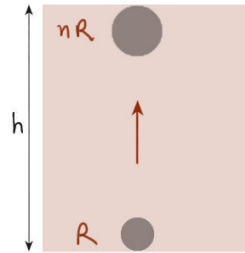
$$p_1 = p_0$$

$$p_2 = p_1 + \rho g h = p_0 + \rho g h$$

$$p_3 = p_2 + \frac{2S}{R} = p_0 + \rho g h + \frac{2S}{R}$$

Youtube / Arihant Senior

2.164. The diameter of a gas bubble formed at the bottom of a pond is $d = 4.0 \mu\text{m}$. When the bubble rises to the surface its diameter increases $n = 1.1$ times. Find how deep is the pond at that spot. The atmospheric pressure is standard, the gas expansion is assumed to be isothermal.



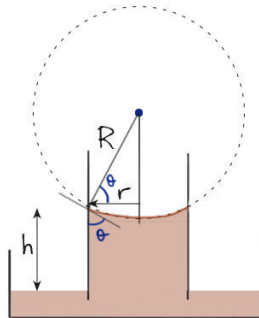
$$P_1 V_1 = P_2 V_2$$

$$\left(P_0 + \frac{2S}{nR}\right) \frac{4}{3} \pi (nR)^3 = \left(P_0 + \rho g h + \frac{2S}{R}\right) \frac{4}{3} \pi R^3$$

$$\Rightarrow h = \frac{P_0 (n^3 - 1) + \frac{2S}{R} (n^3 - 1)}{\rho g}$$

Youtube / Arihant Senior

2.165. Find the difference in height of mercury columns in two communicating vertical capillaries whose diameters are $d_1 = 0.50 \text{ mm}$ and $d_2 = 1.00 \text{ mm}$, if the contact angle $\theta = 138^\circ$.



We know that height of liquid rise in capillary is given by: $h = \frac{2S \cos \theta}{\rho g r}$

where, angle of contact:

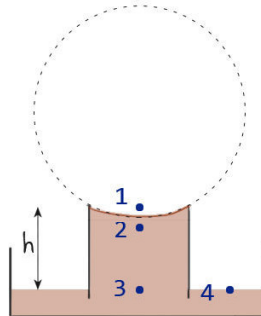
$\cos \theta = \frac{r}{R} \rightarrow$ radius of capillary
 $R \rightarrow$ radius of meniscus

Sol:

$$\therefore h_2 - h_1 = \frac{2S \cos \theta}{\rho g} \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

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2.166. A vertical capillary with inside diameter 0.50 mm is submerged into water so that the length of its part protruding over the water surface is equal to $h = 25$ mm. Find the curvature radius of the meniscus.



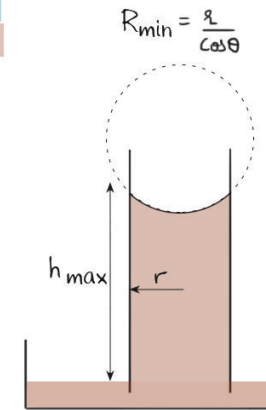
$$P_1 = P_0$$

$$P_2 = P_1 - \frac{2S}{R} = P_0 - \frac{2S}{R}$$

$$P_3 = P_2 + \rho gh = P_0 - \frac{2S}{R} + \rho gh = P_4 = P_0$$

$$\Rightarrow \rho gh = \frac{2S}{R}$$

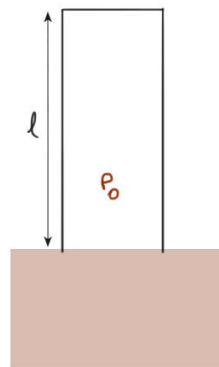
$$\Rightarrow R = \frac{2S}{\rho gh}$$



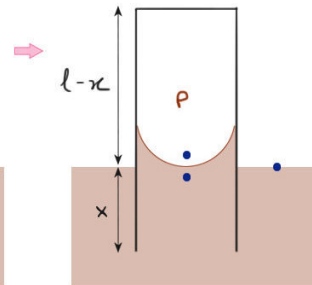
∴ As h increases, R would decrease.
 $\Rightarrow$ When R reaches its minimum value ($= r/\cos\theta$), then h will be maximum.

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2.167. A glass capillary of length $l = 110$ mm and inside diameter $d = 20 \mu\text{m}$ is submerged vertically into water. The upper end of the capillary is sealed. The outside pressure is standard. To what length x has the capillary to be submerged to make the water levels inside and outside the capillary coincide? contact angle for glass-water is zero.



Just after contact



After capillary is submerged to make water levels equal

$$P_1 V_1 = P_2 V_2$$

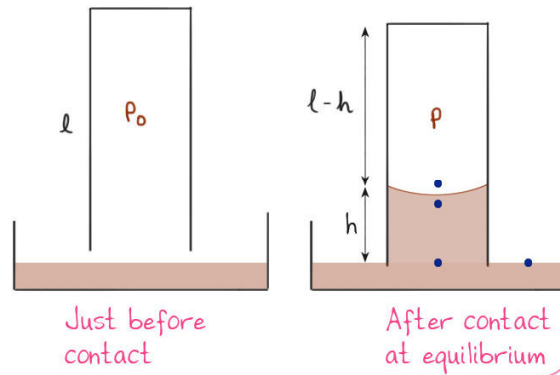
$$\Rightarrow P_0 A l = P A (l-x)$$

$$\Rightarrow P_0 l = \left(P_0 + \frac{2S}{R}\right) (l-x)$$

$$P - \frac{2S}{R} = P_0 \Rightarrow x = \frac{l}{\frac{R P_0}{2S} + 1}$$

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2.168. When a vertical capillary of length l with the sealed upper end was brought in contact with the surface of a liquid, the level of this liquid rose to the height h . The liquid density is ρ , the inside diameter of the capillary is d , the contact angle is θ , the atmospheric pressure is p_0 . Find the surface tension of the liquid.



$$p_1 V_1 = p_2 V_2$$

$$p_0 A l = p A (l-h)$$

$$p_0 l = \left(p_0 + \frac{2S \cos \theta}{r} - \rho g h \right) (l-h)$$

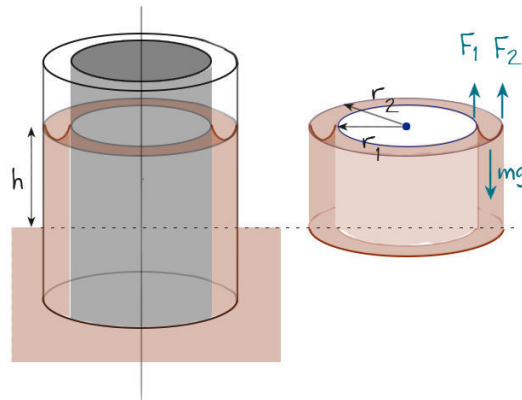
$$\Rightarrow S = \left(\rho g h + \frac{p_0 h}{l-h} \right) \frac{r}{2 \cos \theta}$$

$$p - \frac{2S}{R} + \rho g h = p_0$$

$$\cos \theta = \frac{r}{R}$$

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2.169. A glass rod of diameter $d_1 = 1.5$ mm is inserted symmetrically into a glass capillary with inside diameter $d_2 = 2.0$ mm. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the water rise in the capillary?



$$F_1 + F_2 = mg$$

$$\rightarrow \rho A h$$

$$\Rightarrow (2\pi r_1)S + (2\pi r_2)S = \rho (\pi r_2^2 - \pi r_1^2) h g$$

$$\Rightarrow h = \frac{2\pi S (r_1 + r_2)}{\rho \pi (r_2^2 - r_1^2) g} = \frac{2S}{\rho (r_2 - r_1) g}$$

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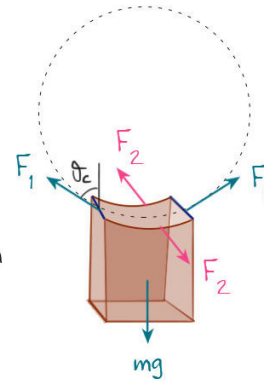
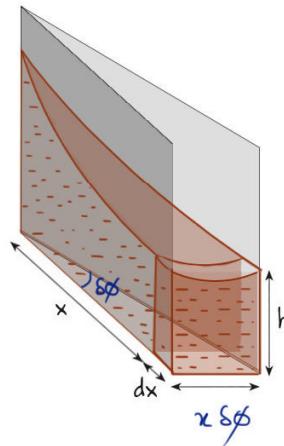
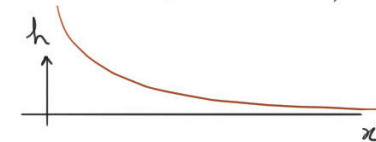
2.170. Two vertical plates submerged partially in a wetting liquid form a wedge with a very small angle $\delta\phi$. The edge of this wedge is vertical. The density of the liquid is ρ , its surface tension is α , the contact angle is θ . Find the height h , to which the liquid rises, as a function of the distance x from the edge.

Force balance on water column of width dx :

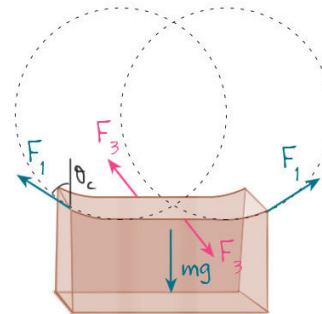
$$2 F_1 \cos \theta_c = mg$$

$$2 (S dx) \cos \theta_c = (\rho \times \delta\phi dx h) g$$

$$\Rightarrow h = \left(\frac{2 S \cos \theta_c}{\rho g \delta\phi} \right) \frac{1}{x}$$



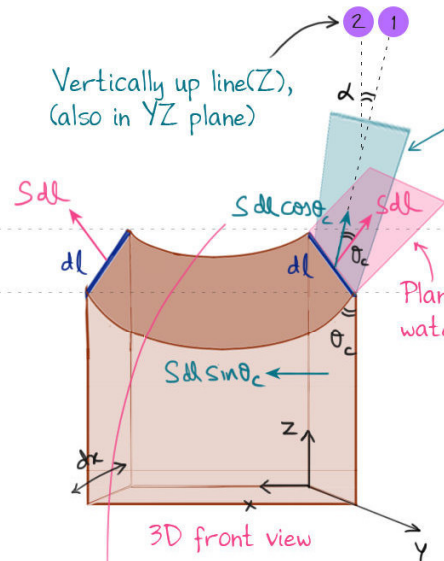
When x is small



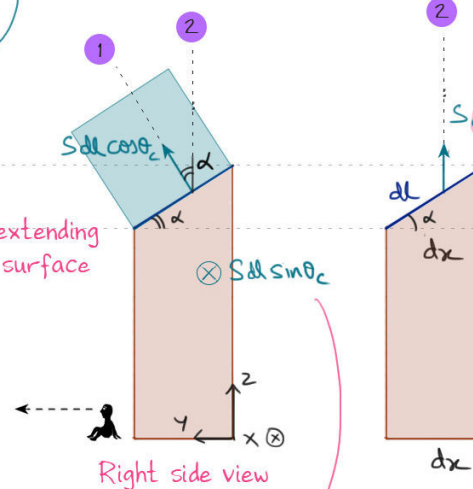
When x is large

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Why upward Force is $S dx \cos \theta_c$



Component of $S dl$ on YZ plane.
Its NOT in Z direction, but along 1
Let's say 1 makes angle α with Z direction.



Other component of $S dl$ in X direction.

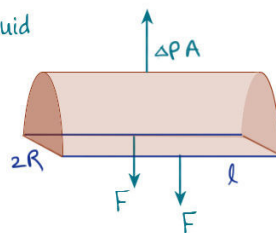
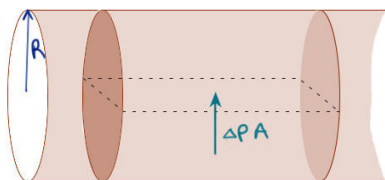
$$S dl \cos \theta_c \cos \alpha = S dx \cos \theta_c$$

$$\therefore \text{Net upward force} = 2 (S dx \cos \theta_c)$$

Youtube / Arihant Senior

2.171. A vertical water jet flows out of a round hole. One of the horizontal sections of the jet has the diameter $d = 2.0 \text{ mm}$ while the other section located $l = 20 \text{ mm}$ lower has the diameter which is $n = 1.5$ times less. Find the volume of the water flowing from the hole each second.

ΔP due to cylindrical shape of liquid



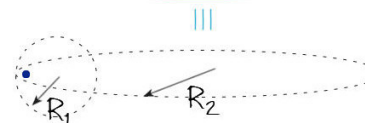
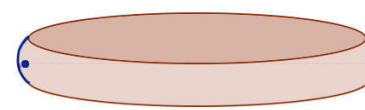
Force balance on semicylindrical shaped liquid:

$$\Delta P A = 2F$$

$$\Delta P \cdot 2Rl = 2Sl$$

$$\Delta P = \frac{S}{R}$$

ΔP due to perpendicular curvatures R_1 and R_2 .



$$\Delta P = \frac{S}{R_1} + \frac{S}{R_2} \quad (\text{Derivation not done here})$$

For spherical: $R_1 = R_2 = R$

For cylindrical: $R_2 = \infty$

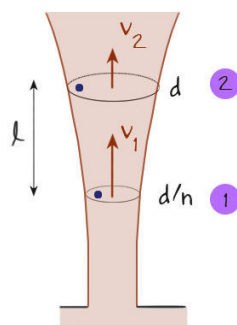
$$\therefore \Delta P = \frac{S}{R} + \frac{S}{R} = \frac{2S}{R}$$

$$\therefore \Delta P = \frac{S}{R} + \frac{S}{\infty} = \frac{S}{R}$$

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We know, $\Delta P = \frac{S}{R}$ (A)

Sol:



By equation of continuity

$$A_1 v_1 = A_2 v_2$$

$$\pi \frac{d^2}{4} v_1 = \pi \frac{d^2}{4n^2} v_2$$

$$v_1 = n^2 v_2 \quad (B)$$

By Bernoulli's equation:

$$P_1 + \rho g h_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2} \rho v_2^2$$

$$\Rightarrow P_0 + \Delta P_1 + \frac{1}{2} \rho v_1^2 = P_0 + \Delta P_2 + \rho g l + \frac{1}{2} \rho v_2^2$$

Using A & B:

$$\Rightarrow \frac{S}{d/2n} + \frac{1}{2} \rho n^4 v_2^2 = \frac{S}{d/2} + \rho g l + \frac{1}{2} \rho v_2^2$$

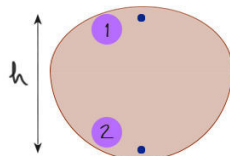
$$\Rightarrow v_2 = \sqrt{\frac{\rho g l - \frac{2S(n-1)}{d}}{\frac{1}{2} \rho (n^4 - 1)}}$$

$$\therefore \text{Flow rate: } A v = \frac{\pi d^2}{4} v_2$$

Youtube / Arihant Senior

2.172. A water drop falls in air with a uniform velocity. Find the difference between the curvature radii of the drop's surface at the upper and lower points of the drop separated by the distance $h = 2.3 \text{ mm}$.

∴ velocity is uniform, pressure difference within the drop exists due to gravity.



$$\therefore P_1 + \rho g h = P_2$$

$$\Rightarrow P_0 + \Delta P_1 + \rho g h = P_0 + \Delta P_2$$

$$\Rightarrow \frac{2S}{R_1} + \rho g h = \frac{2S}{R_2}$$

$$\Rightarrow \rho g h = \frac{2S(R_1 - R_2)}{R_1 R_2} \approx R^2 = \left(\frac{h}{2}\right)^2 = \frac{h^2}{4}$$

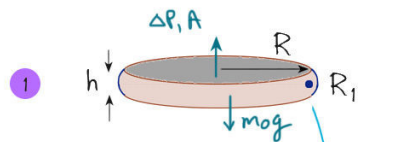
$$\Rightarrow R_1 - R_2 = \frac{\rho g h}{2S} \frac{h^2}{4} = \frac{\rho g h^3}{8S}$$

Note: If it were free fall (acceleration g), $P_1 = P_2$ and drop will be perfectly spherical.

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2.173. A mercury drop shaped as a round tablet of radius R and thickness h is located between two horizontal glass plates. Assuming that $h \ll R$, find the mass m of a weight which has to be placed on the upper plate to diminish the distance between the plates n -times. The contact angle equals θ . Calculate m if $R = 2.0 \text{ cm}$, $h = 0.38 \text{ mm}$, $n = 2.0$, and $\theta = 135^\circ$.

Let m_0 be the mass of top plate



ΔP is due to perpendicular curvatures R_1 and R_2

This ΔP supports the overall mass m_0 (of plate and Hg) at table top.

$$\textcircled{1} \Delta P = \frac{S}{R_1} + \frac{S}{R_2} \approx \frac{S}{R_1} \text{ and } \sin\left(\theta - \frac{\pi}{2}\right) = \frac{h/2}{R_1} \text{ and } \Delta P \pi R^2 = m_0 g$$

(From 2.171)

$$\text{From above 3 equations, } m_0 = - \frac{2\pi R^2 S \cos\theta}{gh}$$

$$\textcircled{2} \text{ Similarly: } m_0 + m = - \frac{2\pi R_2^2 S \cos\theta}{g h/n}$$

By V conservation, $\pi R^2 h = \pi R_3^2 \frac{h}{n}$

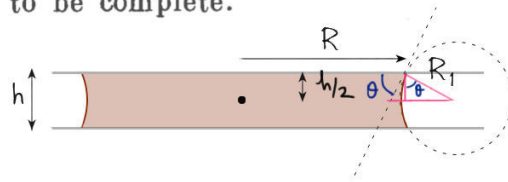
Subtracting the two:

$$\Rightarrow m = - \frac{2\pi R^2 S \cos\theta}{gh} (n^2 - 1)$$

$$R_3 = R\sqrt{n}$$

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2.174. Find the attraction force between two parallel glass plates, separated by a distance $h = 0.10$ mm, after a water drop of mass $m = 70$ mg was introduced between them. The wetting is assumed to be complete.



$\therefore \Delta P \text{ is } -ve \Rightarrow \text{Force is attractive}$

$\Delta P = \frac{S}{R} - \frac{S}{R_1} \approx 0$ and $\cos \theta = \frac{h/2}{R_1}$ and $\frac{4}{3} \pi R^2 h = m$

(From 2.171)

$\therefore \text{Required attraction force} = \Delta P \cdot A = \frac{S}{R_1} \cdot \pi R^2$

$= \frac{S \cos \theta}{h/2} \cdot \pi R^2 = \frac{2 \pi R^2 S \cos \theta}{h}$

$= \frac{2 S \cos \theta}{3 h^2} m$

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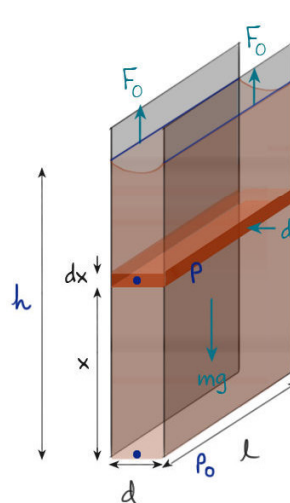
2.175. Two glass discs of radius $R = 5.0$ cm were wetted with water and put together so that the thickness of the water layer between them was $h = 1.9$ μm . Assuming the wetting to be complete, find the force that has to be applied at right angles to the plates in order to pull them apart.

From previous 2.174,

Attraction force that needs to be overcome = $\frac{2 \pi R^2 S \cos \theta}{h}$

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2.176. Two vertical parallel glass plates are partially submerged in water. The distance between the plates is $d = 0.10$ mm, and their width is $l = 12$ cm. Assuming that the water between the plates does not reach the upper edges of the plates and that the wetting is complete, find the force of their mutual attraction.



dF on plate element, due to pressure difference is:

$$\begin{aligned} dF &= \Delta P \cdot A \\ &= \rho g x \cdot l \cdot dx \\ \Rightarrow F &= \rho g l \int_0^h x \cdot dx \\ &= \rho g l \cdot \frac{1}{2} S^2 \omega^2 \theta \\ &= \frac{2 l S^2 \omega^2 \theta}{\rho g d^2} // \end{aligned}$$

Finding h

$$\begin{aligned} mg &= 2 F_0 \\ \Rightarrow (\rho d / h) g &= 2 S \omega^2 \theta \\ \Rightarrow h &= \frac{2 S \omega^2 \theta}{\rho g d} // \end{aligned}$$

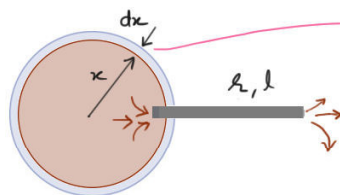
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2.177. Find the lifetime of a soap bubble of radius R connected with the atmosphere through a capillary of length l and inside radius r . The surface tension is α , the viscosity coefficient of the gas is η .

Flow rate through capillary is given by poiseuille's equation:

$$\frac{dV}{dt} = \frac{\pi r^4 \Delta P}{8 \eta l}$$

Sol:



Let bubble radius changes by dx in dt time.

$$\begin{aligned} \frac{dV}{dt} &= -4 \pi r^2 \frac{dx}{dt} = \frac{\pi r^4}{8 \eta l} \left(\frac{4 S}{x} \right) \\ \Rightarrow \int_R^0 x^3 dx &= - \frac{r^4 S}{8 \eta l} \int_0^T dt \end{aligned}$$

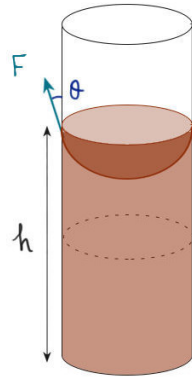
$$\Rightarrow T = \frac{2 \eta l}{r^4 S} R^4 //$$

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2.178. A vertical capillary is brought in contact with the water surface. What amount of heat is liberated while the water rises along the capillary? The wetting is assumed to be complete, the surface tension equals α .

We know, in capillary:

$$h = \frac{2S \cos \theta}{\rho g r}$$



By conservation of energy:

$$W_s + W_g = \Delta KE + \text{Heat}$$

$$\Rightarrow \text{Heat} = F \cos \theta \cdot h - \frac{mgh}{2}$$

$$= (S \cdot 2\pi r) \left(\frac{\rho g r h}{2S} \right) h - \frac{mgh}{2}$$

$$= \underbrace{S \pi r^2 h}_{m} \cdot g \cdot h - \frac{mgh}{2}$$

∴ Heat lost is equal to gravitational PE of liquid column.

$$= \frac{mgh}{2} // = \frac{\rho g \pi r^2 h^2}{2} // = \frac{\rho g \pi r^2 \cdot \frac{4S^2 \cos^2 \theta}{\rho^2 g^2 r^2}}{2} = \frac{2\pi S^2 \cos^2 \theta}{\rho g}$$

Youtube / Arihant Senior

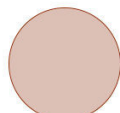
2.179. Find the free energy of the surface layer of

(a) a mercury droplet of diameter $d = 1.4$ mm;

(b) a soap bubble of diameter $d = 6.0$ mm if the surface tension of the soap water solution is equal to $\alpha = 45$ mN/m.

Surface energy is given by: $E_s = S \cdot A$

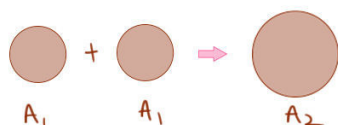
Total surface area

A  $E_s = S \cdot A = S \cdot \frac{4\pi d^2}{4} = S \cdot \pi d^2 //$

B  $E_s = S \cdot A = S \cdot 2 \left(\frac{4\pi d^2}{4} \right) = 2S\pi d^2 //$

∴ bubble has two surfaces, both of area $\frac{4\pi d^2}{4}$

2.180. Find the increment of the free energy of the surface layer when two identical mercury droplets, each of diameter $d = 1.5$ mm, merge isothermally.



Volume conservation:

$$2 \cdot \frac{4}{3} \pi R_1^3 = \frac{4}{3} \pi R_2^3$$

$$\Rightarrow R_2 = 2^{1/3} R_1$$

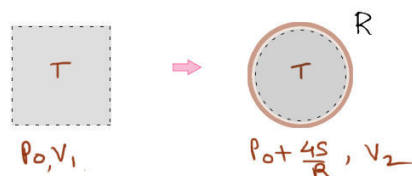
Increment in surface energy:

$$\begin{aligned} \Delta E &= S A_2 - 2 S A_1 \\ &= S (A_2 - 2 A_1) \\ &= S (4\pi R_2^2 - 2 \cdot 4\pi R_1^2) \\ &= 4\pi S (2^{2/3} - 2) \\ &= -4\pi S (2 - 2^{2/3}) = -1.43 \text{ MJ} \end{aligned}$$

Surface energy decreases on merging of droplets.

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2.181. Find the work to be performed in order to blow a soap bubble of radius R if the outside air pressure is equal to p_0 and the surface tension of the soap water solution is equal to α .



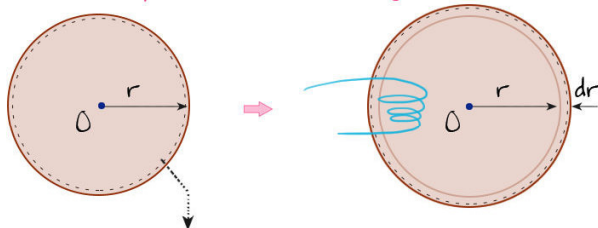
Assuming isothermal process, internal energy of gas remains constant.
So Total external work gets used up as:

$$\begin{aligned} W_{\text{external}} &= E_{\text{film formation}} + W_{\text{isothermal compression of gas}} \\ &= S \cdot 2(4\pi R^2) + p_2 V_2 \ln \frac{p_2}{p_1} \\ &= 8\pi S R^2 + \left(p_0 + \frac{4S}{R}\right) \left(\frac{4}{3}\pi R^3\right) \ln \left(\frac{p_0 + \frac{4S}{R}}{p_0}\right) \end{aligned}$$

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2.182. A soap bubble of radius r is inflated with an ideal gas. The atmospheric pressure is p_0 , the surface tension of the soap water solution is α . Find the difference between the molar heat capacity of the gas during its heating inside the bubble and the molar heat capacity of the gas under constant pressure, $C - C_p$.

Lets say the radius of bubble increases by dr when temperature increases by dT



Consider gas as the System

- We don't need to bother with Work done by atmosphere or Energy stored in film anymore.

For the gas:

$$dQ = dU + PdV$$

$$\Rightarrow C = C_v + \frac{PdV}{n_0 dT} \quad (\text{For small change, } P \text{ does not change significantly})$$

$$= C_v + \left(p_0 + \frac{4\alpha}{r}\right) \frac{4\pi r^2 dr}{n_0 dT}$$

$$= C_v + \left(p_0 + \frac{4\alpha}{r}\right) \frac{R}{p_0 + \frac{8\alpha}{3r}}$$

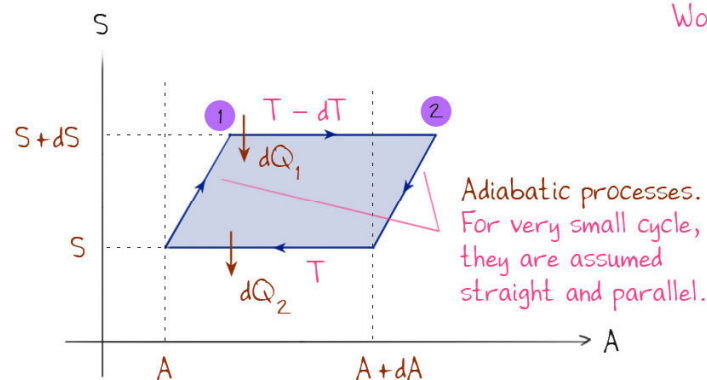
$$PV = n_0 RT$$

$$\Rightarrow \left(p_0 + \frac{4\alpha}{r}\right) \frac{4}{3} \pi r^3 = n_0 RT$$

$$\text{diff.} \Rightarrow \frac{dr}{dT} = \frac{n_0 R}{4\pi r^2 \left(p_0 + \frac{8\alpha}{3r}\right)}$$

Youtube/Arihant Senior

2.183. Considering the Carnot cycle as applied to a liquid film, show that in an isothermal process the amount of heat required for the formation of a unit area of the surface layer is equal to $q = -T \cdot d\alpha/dT$, where $d\alpha/dT$ is the temperature derivative of the surface tension.



$$\text{Work done on film} = \int S dA$$

$$= \text{Area under } SA \text{ graph}$$

$$= dS dA \quad (\text{for cyclic process})$$

For small carnot cycle:

$$\eta = \frac{dT}{T} = \frac{\text{Work done by film}}{dQ_1}$$

$$\Rightarrow \frac{dT}{T} = -\frac{dS dA}{dQ_1}$$

$$\Rightarrow \frac{dQ_1}{dA} = -T \frac{dS}{dT}$$

Note: When Area of film is increased adiabatically, Temperature reduces (derivation beyond our scope).

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2.184. The surface of a soap film was increased isothermally by $\Delta\sigma$ at a temperature T . Knowing the surface tension of the soap water solution σ and the temperature coefficient $d\sigma/dT$, find the increment

- of the entropy of the film's surface layer;
- of the internal energy of the surface layer.

We know, $\frac{dQ}{dA} = -T \frac{dS}{dT}$

A Entropy = $\frac{dQ}{T} = -dA \frac{dS}{dT}$

$\nearrow 2\Delta\sigma$ ($\because$ soap film has two layers)

$$= -2\Delta\sigma \frac{dS}{dT}$$

B $dU = dQ + W_{\text{on film}}$

$S\Delta A$

$$= -T 2\Delta\sigma \frac{dS}{dT} + S(2\Delta\sigma) = 2\Delta\sigma \left(S - T \frac{dS}{dT} \right)$$

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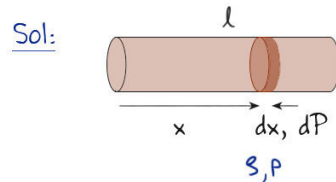
2.246. Making use of Poiseuille's equation (1.7d), find the mass μ of gas flowing per unit time through the pipe of length l and radius r if constant pressures p_1 and p_2 are maintained at its ends.

Poiseuille's equation: $\frac{dV}{dt} = \frac{\pi r^4}{8\eta} \frac{\Delta p}{l}$

If it were liquid, answer would be: $\frac{dm}{dt} = \frac{\rho dV}{dt} = \frac{\rho}{8\eta} \frac{\pi r^4 (p_1 - p_2)}{l}$

But in gas flow, ρ is not constant.

So we write Poiseuille's equation for a small gaseous element where ρ can be assumed fixed.



$$\frac{dm}{dt} = \frac{\rho \pi r^4}{8\eta} \frac{dp}{dx}$$

constant across tube $\frac{\rho M}{RT}$

$$\Rightarrow \frac{dm}{dt} \int_0^l dx = \frac{M}{RT} \frac{\pi r^4}{8\eta} \int_{p_1}^{p_2} p dp \Rightarrow \frac{dm}{dt} = \frac{M}{RT} \frac{\pi r^4}{8\eta} \frac{(p_2^2 - p_1^2)}{l}$$

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2.247. One end of a rod, enclosed in a thermally insulating sheath, is kept at a temperature T_1 while the other, at T_2 . The rod is composed of two sections whose lengths are l_1 and l_2 and heat conductivity coefficients k_1 and k_2 . Find the temperature of the interface.

We know, heat current:

$$\frac{dq}{dt} = kA \frac{\Delta T}{l}$$



Heat current is constant.

$$\therefore k_1 A \frac{(T_1 - T)}{l_1} = k_2 A \frac{(T - T_2)}{l_2}$$

$$\Rightarrow T = \frac{\frac{k_1 T_1}{l_1} + \frac{k_2 T_2}{l_2}}{\frac{k_1}{l_1} + \frac{k_2}{l_2}} //$$

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2.248. Two rods whose lengths are l_1 and l_2 and heat conductivity coefficients k_1 and k_2 are placed end to end. Find the heat conductivity coefficient of a uniform rod of length $l_1 + l_2$ whose conductivity is the same as that of the system of these two rods. The lateral surfaces of the rods are assumed to be thermally insulated.



$$\text{Heat resistance} = \frac{l}{kA}$$

|||



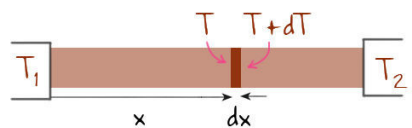
For series combination of heat resistors: $R_{eq} = R_1 + R_2$

$$\Rightarrow \frac{l_1 + l_2}{k_{eq} A} = \frac{l_1}{k_1 A} + \frac{l_2}{k_2 A}$$

$$\Rightarrow k_{eq} = \frac{l_1 + l_2}{\frac{l_1}{k_1} + \frac{l_2}{k_2}} //$$

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2.249. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as $k = \alpha/T$, where α is a constant. The ends of the rod are kept at temperatures T_1 and T_2 . Find the function $T(x)$, where x is the distance from the end whose temperature is T_1 , and the heat flow density.



$$\frac{dQ}{dt} = kA \left[\frac{T - (T+dT)}{dx} \right]$$

Constant $\Rightarrow \frac{dQ}{dt} dx = \frac{\alpha}{T} A (-dT)$

$$\Rightarrow \frac{dQ}{dt} \int_0^l dx = -A\alpha \int_{T_1}^{T_2} \frac{dT}{T}$$

$$\Rightarrow \frac{dQ}{dt} = -\frac{A\alpha}{l} \ln \frac{T_2}{T_1}$$

$$\Rightarrow T = T_1 e^{-\frac{x}{A\alpha} \frac{dQ}{dt}} = T_1 \left(\frac{T_2}{T_1} \right)^{x/l}$$

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2.250. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area S and fairly low heat conductivity k . The whole system is thermally insulated from the environment. At a moment $t = 0$ the temperature difference between the two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time.

In time dt , dQ = heat lost by 1 = heat gained by 2

$$\Rightarrow kA \frac{\Delta T}{l} dt = -C_1 dT_1 = C_2 dT_2 \quad (1)$$

Also, $d\Delta T = \Delta T_f - \Delta T_i$

$$= [(T_1 + dT_1) - (T_2 + dT_2)] - (T_1 - T_2)$$

$$\Rightarrow d\Delta T = dT_1 - dT_2 \quad (2)$$

From 1,2:

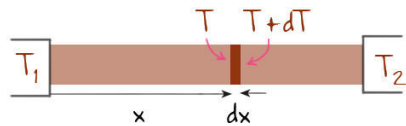
$$d\Delta T = -\frac{kA}{l} \Delta T dt \left(\frac{1}{C_2} + \frac{1}{C_1} \right)$$

$$\Rightarrow \int_{\Delta T_0}^{\Delta T} \frac{d\Delta T}{\Delta T} = -\frac{kA}{l} \left(\frac{1}{C_1} + \frac{1}{C_2} \right) \int_0^t dt$$

$$\Rightarrow \Delta T = \Delta T_0 e^{-\frac{kA}{l} \left(\frac{1}{C_1} + \frac{1}{C_2} \right) t}$$

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2.251. Find the temperature distribution in a substance placed between two parallel plates kept at temperatures T_1 and T_2 . The plate separation is equal to l , the heat conductivity coefficient of the substance $k \propto \sqrt{T}$.



$$\frac{dQ}{dt} = kA \left(-\frac{dT}{dx} \right)$$

$$\Rightarrow \frac{dQ}{dt} = -C\sqrt{T} A \frac{dT}{dx}$$

$$\Rightarrow \frac{dQ}{dt} \int_0^l dx = -CA \int_{T_1}^{T_2} \sqrt{T} dT$$

$x: 0 \rightarrow x$
 $T: T_1 \rightarrow T$

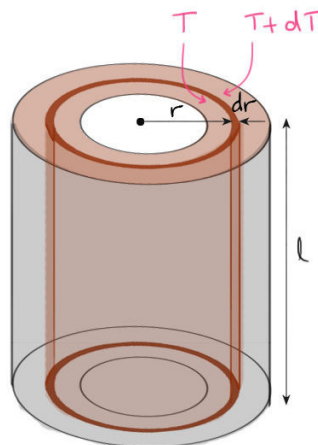
$$\frac{dQ}{dt} l = -\frac{2}{3} CA (T_2^{3/2} - T_1^{3/2}) \quad (1)$$

$$\frac{dQ}{dt} x = -\frac{2}{3} CA (T^{3/2} - T_1^{3/2}) \quad (2)$$

Dividing 1 and 2: $T = T_1 \left[1 + \frac{x}{l} \left\{ \left(\frac{T_2}{T_1} \right)^{3/2} - 1 \right\} \right]^{2/3}$

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2.254. Find the temperature distribution in the space between two coaxial cylinders of radii R_1 and R_2 filled with a uniform heat conducting substance if the temperatures of the cylinders are constant and are equal to T_1 and T_2 respectively.



$$\frac{dQ}{dt} = k(2\pi r l) \left(-\frac{dT}{dr} \right)$$

$$\Rightarrow \frac{dQ}{dt} \int_{R_1}^{R_2} \frac{dr}{r} = -2k\pi l \int_{T_1}^{T_2} dT$$

$r: R_1 \rightarrow r$
 $T: T_1 \rightarrow T$

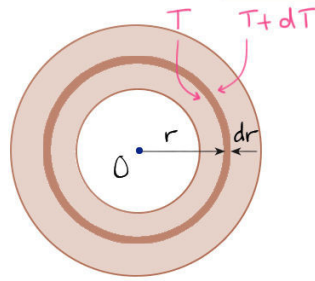
$$\frac{dQ}{dt} \ln \frac{R_2}{R_1} = -2k\pi l (T_2 - T_1) \quad (1)$$

$$\frac{dQ}{dt} \ln \frac{r}{R_1} = -2k\pi l (T - T_1) \quad (2)$$

Dividing 1 and 2: $T = T_1 + (T_2 - T_1) \frac{\ln r/R_1}{\ln R_2/R_1}$

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2.255. Solve the foregoing problem for the case of two concentric spheres of radii R_1 and R_2 and temperatures T_1 and T_2 .



$$\frac{dQ}{dt} = k 4\pi r^2 \left(-\frac{dT}{dr}\right)$$

$$\Rightarrow \frac{dQ}{dt} \int_{R_1}^{R_2} \frac{dr}{r^2} = -4k\pi \int_{T_1}^{T_2} dT$$

$r: R_1 \rightarrow R_2$
 $T: T_1 \rightarrow T_2$

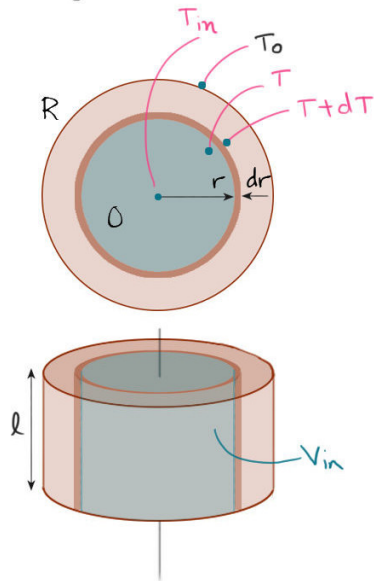
$$\frac{dQ}{dt} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = -4k\pi (T_2 - T_1) \quad (1)$$

$$\frac{dQ}{dt} \left(\frac{1}{R_1} - \frac{1}{r} \right) = -4k\pi (T - T_1) \quad (2)$$

Dividing 1 and 2: $T = T_1 + (T_2 - T_1) \frac{\left(\frac{1}{R_1} - \frac{1}{r} \right)}{\left(\frac{1}{R_1} - \frac{1}{R_2} \right)}$

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2.256. A constant electric current flows along a uniform wire with cross-sectional radius R and heat conductivity coefficient k . A unit volume of the wire generates a thermal power w . Find the temperature distribution across the wire provided the steady-state temperature at the wire surface is equal to T_0 .



Heat current across a thin element of thickness dr is given by:

$$\frac{dQ}{dt} = V_{in} w = \pi r^2 l w = k A \left(-\frac{dT}{dr}\right) = -k 2\pi r l \frac{dT}{dr}$$

$$\Rightarrow \pi r^2 l w = -k 2\pi r l \frac{dT}{dr} \quad \frac{w R^2}{2} = -2k (T_0 - T_{in}) \quad (1)$$

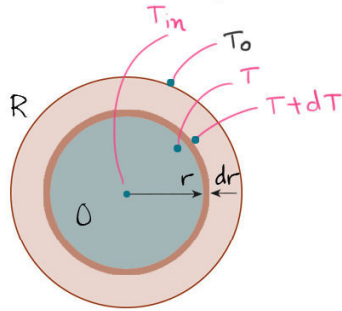
$$\Rightarrow \int_0^R r dr = -2k \int_{T_{in}}^{T_0} dT \quad \frac{w R^2}{2} = -2k (T - T_{in}) \quad (2)$$

$r: 0 \rightarrow R$
 $T: T_{in} \rightarrow T$

Subtracting 1 and 2: $T = T_0 + \frac{w (R^2 - r^2)}{4k}$

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2.257. The thermal power of density w is generated uniformly inside a uniform sphere of radius R and heat conductivity coefficient k . Find the temperature distribution in the sphere provided the steady-state temperature at its surface is equal to T_0 .



$$\frac{dQ}{dt} = V_{in} \omega = \frac{4}{3} \pi r^3 \omega = k A \left(\frac{dT}{dr} \right) = -k 4\pi r^2 \frac{dT}{dr}$$

$$\Rightarrow \frac{4}{3} \pi r^3 \omega = -k 4\pi r^2 \frac{dT}{dr}$$

$$\Rightarrow \frac{\omega}{3} \int_0^R r^3 dr = -k \int_{T_{in}}^{T_0} dT$$

$$\begin{aligned} \frac{\omega R^2}{6} &= -k(T_0 - T_{in}) \quad (1) \\ \frac{\omega r^2}{6} &= -k(T - T_{in}) \quad (2) \end{aligned}$$

$r: 0 \rightarrow R$
 $T: T_{in} \rightarrow T$

Subtracting 1 and 2: $T = T_0 + \frac{\omega(R^2 - r^2)}{6k}$ //

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4.1. A point oscillates along the x axis according to the law $x = a \cos(\omega t - \pi/4)$. Draw the approximate plots

(a) of displacement x , velocity projection v_x , and acceleration projection w_x as functions of time t ;

(b) velocity projection v_x and acceleration projection w_x as functions of the coordinate x .

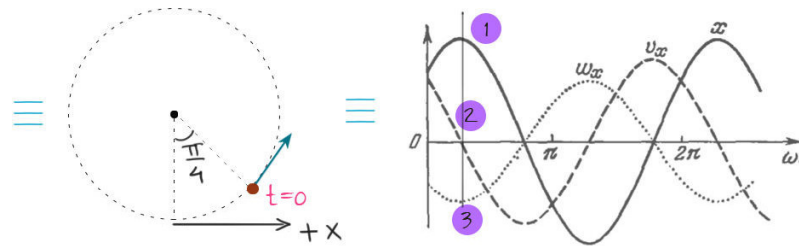
A $x = a \cos(\omega t - \frac{\pi}{4})$

$\because \cos \theta = \sin(\frac{\pi}{2} + \theta)$

$\approx x = a \sin(\omega t + \frac{\pi}{4}) \quad (1)$

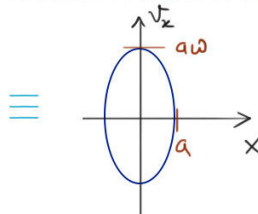
$\Rightarrow v_x = \frac{dx}{dt} = a\omega \cos(\omega t + \frac{\pi}{4}) \quad (2)$

$\Rightarrow w_x = \frac{d^2x}{dt^2} = -a\omega^2 \sin(\omega t + \frac{\pi}{4}) \quad (3)$



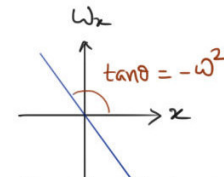
B From 1,2:

$$\frac{x^2}{a^2} + \frac{v_x^2}{a^2 \omega^2} = 1$$



From 1,3:

$$w_x = -\omega^2 x$$



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4.2. A point moves along the x axis according to the law $x = a \sin^2(\omega t - \pi/4)$. Find:

- (a) the amplitude and period of oscillations; draw the plot $x(t)$;
 (b) the velocity projection v_x as a function of the coordinate x ; draw the plot $v_x(x)$.

A $x = a \sin^2(\omega t - \frac{\pi}{4})$

$\approx x = \frac{a}{2} (1 - \cos 2(\omega t - \frac{\pi}{4}))$

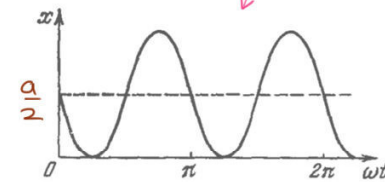
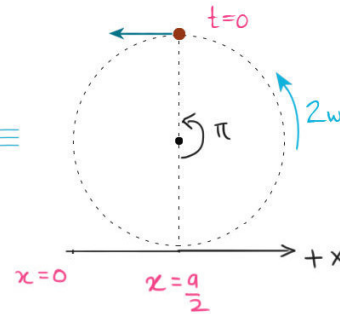
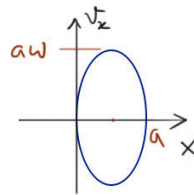
$\approx x = \frac{a}{2} + \frac{a}{2} \sin(2\omega t + \pi)$ ①

↓ diff

B $v_x = a\omega \cos(2\omega t + \pi)$ ②

From 1,2:

$\frac{(x - a/2)^2}{(a/2)^2} + \frac{v_x^2}{a^2\omega^2} = 1$



∴ repeats at $\omega t = \pi$

$T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$

$\Rightarrow \omega T = \pi$

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4.3. A particle performs harmonic oscillations along the x axis about the equilibrium position $x = 0$. The oscillation frequency is $\omega = 4.00 \text{ s}^{-1}$. At a certain moment of time the particle has a coordinate $x_0 = 25.0 \text{ cm}$ and its velocity is equal to $v_{x0} = 100 \text{ cm/s}$. Find the coordinate x and the velocity v_x of the particle $t = 2.40 \text{ s}$ after that moment.

From initial conditions given we will find equation of shm at $t=0$ (lets say, when its at x_0, v_0). Then for $t=2.4$ we can find x and v .

Let, $x - x_m = A \sin(\omega t + \phi)$ $\xrightarrow{t=0}$ $x_0 = A \sin \phi$ ①
 $v_x = A\omega \cos(\omega t + \phi)$ $\xrightarrow{t=0}$ $v_0 = A\omega \cos \phi$ ②

From 1,2: $A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$, $\phi = \tan^{-1} \frac{x_0 \omega}{v_0}$

∴ At given time t , $x = A \sin(\omega t + \phi)$
 $v = A\omega \cos(\omega t + \phi)$

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4.4. Find the angular frequency and the amplitude of harmonic oscillations of a particle if at distances x_1 and x_2 from the equilibrium position its velocity equals v_1 and v_2 respectively.

$$v^2 = \omega^2 (A^2 - x^2)$$

$$v_1^2 = \omega^2 (A^2 - x_1^2) \quad (1)$$

$$v_2^2 = \omega^2 (A^2 - x_2^2) \quad (2)$$

From 1,2:

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}, \quad A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$$

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4.5. A point performs harmonic oscillations along a straight line with a period $T = 0.60$ s and an amplitude $a = 10.0$ cm. Find the mean velocity of the point averaged over the time interval during which it travels a distance $a/2$, starting from

- the extreme position;
- the equilibrium position.

$$\text{Let } x = A \sin \omega t$$

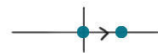
B From equilibrium to $a/2$:

$$\frac{A}{2} = A \sin \omega t_1$$

$$\frac{\pi}{6} = \omega t_1 \quad \left(\omega = \frac{2\pi}{T} \right)$$

$$\Rightarrow t_1 = \frac{T}{12}$$

$$\Rightarrow v_{av} = \frac{a/2}{T/12} = \frac{6a}{T}$$

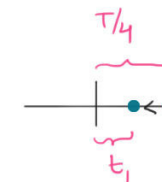


A From $a/2$ to extreme:

$t_2 = t$ from equilibrium to extreme
 $- t$ from equilibrium to $a/2$.

$$t_2 = T/4 - t_1 = \frac{T}{4} - \frac{T}{12} = \frac{T}{6}$$

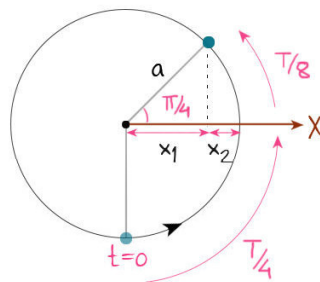
$$\Rightarrow v_{av} = \frac{a/2}{T/6} = \frac{3a}{T}$$



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4.6. At the moment $t = 0$ a point starts oscillating along the x axis according to the law $x = a \sin \omega t$. Find:

- (a) the mean value of its velocity vector projection $\langle v_x \rangle$; $\rightarrow$ Average v_x
 (b) the modulus of the mean velocity vector $|\langle \vec{v} \rangle|$; $\rightarrow$ Av. $|\vec{v}|$
 (c) the mean value of the velocity modulus $\langle v \rangle$ averaged over $3/8$ of the period after the start. $\rightarrow$ Av. speed



$$\Delta t = \frac{3T}{8} = \frac{3}{8} \cdot \left(\frac{2\pi}{\omega} \right) = \frac{3\pi}{4\omega}$$

A Average $v_x = \frac{x_2 - x_1}{\Delta t} = \frac{a/\sqrt{2} - 0}{3\pi/4\omega} = \frac{2\sqrt{2} a \omega}{3\pi}$

B Av. $|\vec{v}| = \left| \frac{\vec{r}_2 - \vec{r}_1}{\Delta t} \right| = \left| \frac{2\sqrt{2} a \omega}{3\pi} \hat{i} \right| = \frac{2\sqrt{2} a \omega}{3\pi}$

C Av. speed = $\frac{a + x_2}{\Delta t}$

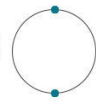
$$= \frac{a + (a - a/\sqrt{2})}{\Delta t} = \frac{a + (a - a/\sqrt{2})}{\Delta t} = \frac{2(4 - \sqrt{2}) a \omega}{3\pi}$$

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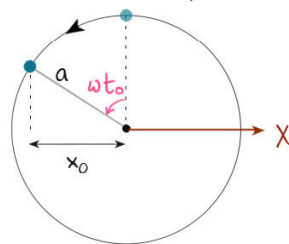
4.7. A particle moves along the x axis according to the law $x = a \cos \omega t$. Find the distance that the particle covers during the time interval from $t = 0$ to t .

Lets say: $t = n(\frac{T}{4}) + t_0$ $\{t_0 < \frac{T}{4}\}$ or $t_0 = t - n\frac{T}{4} = t - \frac{n\pi}{2\omega}$ ①

Case 1 n is odd

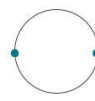


Its at mean position + t_0

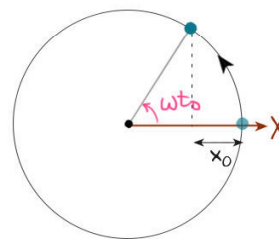


$$\begin{aligned} \therefore S &= na + x_0 \\ &= na + a \sin \omega t_0 \\ &= na + a \sin \left(\omega t - \frac{n\pi}{2} \right) \end{aligned}$$

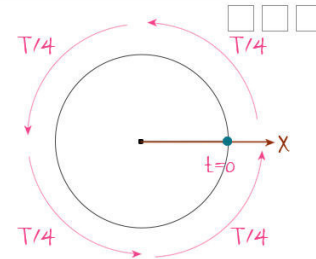
Case 2 n is even



Its at extreme position + t_0



$$\begin{aligned} \therefore S &= na + x_0 \\ &= na + a - a \cos \omega t_0 \\ &= a \left[n+1 - \cos \left(\omega t - \frac{n\pi}{2} \right) \right] \end{aligned}$$



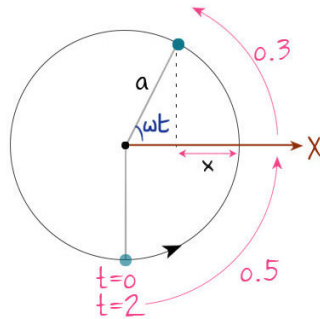
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4.8. At the moment $t = 0$ a particle starts moving along the x axis so that its velocity projection varies as $v_x = 35 \cos \pi t$ cm/s, where t is expressed in seconds. Find the distance that this particle covers during $t = 2.80$ s after the start.

$$v = 35 \cos \pi t$$

$$35 = \omega a \Rightarrow a = \frac{35}{\omega} = \frac{35}{\pi}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \text{ s}$$



$$S = \frac{4a}{2} + \frac{a}{0.5} + \frac{\pi}{0.3}$$

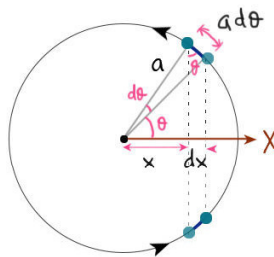
$$\therefore S = 4a + a + a(1 - \cos \omega t)$$

$$= a \left(6 - \cos \frac{3\pi}{10} \right)$$

$$= \frac{35}{\pi} (6 - \cos 54^\circ) = 59.5 \text{ cm} //$$

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4.9. A particle performs harmonic oscillations along the x axis according to the law $x = a \cos \omega t$. Assuming the probability P of the particle to fall within an interval from $-a$ to $+a$ to be equal to unity, find how the probability density dP/dx depends on x . Here dP denotes the probability of the particle falling within an interval from x to $x + dx$. Plot dP/dx as a function of x .



Method 1

Probability of being in dx is given by:

$$dP = \frac{2(a d\theta)}{\text{circumference}} = \frac{2a d\theta}{2\pi a} = \frac{d\theta}{\pi} \quad (1)$$

Also, $(a d\theta) \sin \theta = dx \quad (2)$

$$\sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{x^2}{a^2}}$$

Eliminating θ from 1,2:

$$dP = \frac{dx}{\pi \sqrt{a^2 - x^2}} \Rightarrow \frac{dP}{dx} = \frac{1}{\pi \sqrt{a^2 - x^2}} //$$

Method 2

Probability of being in dx is given by:

$$dP = \frac{\text{Time spent in } dx}{T} = \frac{2 \left(\frac{dx}{v_x} \right)}{T}$$

$$\Rightarrow \frac{dP}{dx} = \frac{2}{v_x T} = \frac{2}{\omega \sqrt{a^2 - x^2} \frac{2\pi}{\omega}}$$

$$= \frac{1}{\pi \sqrt{a^2 - x^2}} //$$

Youtube / Arihant Senior

4.10. Using graphical means, find an amplitude a of oscillations resulting from the superposition of the following oscillations of the same direction:

(a) $x_1 = 3.0 \cos(\omega t + \pi/3)$, $x_2 = 8.0 \sin(\omega t + \pi/6)$;

(b) $x_1 = 3.0 \cos \omega t$, $x_2 = 5.0 \cos(\omega t + \pi/4)$, $x_3 = 6.0 \sin \omega t$.

A

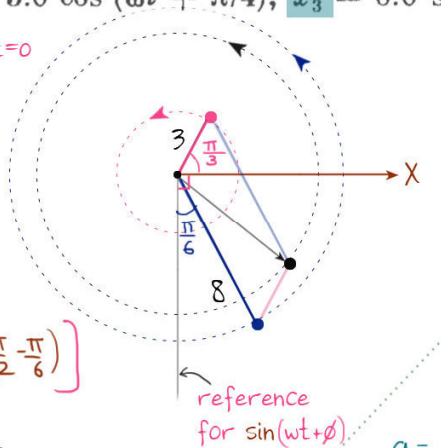
$$x_1 = 3 \sin \left[\frac{\pi}{2} + (\omega t + \frac{\pi}{3}) \right]$$

$$x_2 = 8 \sin(\omega t + \frac{\pi}{6})$$

$$\therefore a^2 = 3^2 + 8^2 + 2 \cdot 3 \cdot 8 \cos \left[\frac{\pi}{3} + \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \right]$$

$$\Rightarrow a = \sqrt{73 - 48 \left(\frac{1}{2} \right)} = \sqrt{49} = 7 //$$

At $t=0$

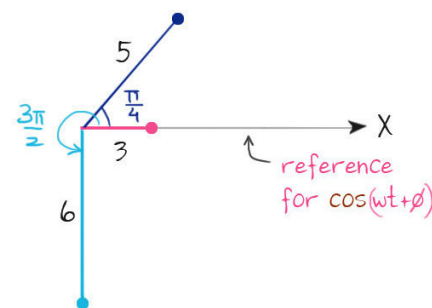


B

$$x_1 = 3 \cos \omega t$$

$$x_2 = 5 \cos(\omega t + \frac{\pi}{4})$$

$$x_3 = 6 \cos(\frac{3\pi}{2} + \omega t)$$



$$a = \sqrt{(3 + 5 \cos \frac{\pi}{4})^2 + (6 - 5 \sin \frac{\pi}{4})^2}$$

$$= \sqrt{48.7} \approx 7 //$$

Youtube / Arihant Senior

4.11. A point participates simultaneously in two harmonic oscillations of the same direction: $x_1 = a \cos \omega t$ and $x_2 = a \cos 2\omega t$. Find the maximum velocity of the point.

$$x = x_1 + x_2 = a \cos \omega t + a \cos 2\omega t$$

$$v = \frac{dx}{dt} = -a\omega \sin \omega t - 2a\omega \sin 2\omega t \quad (1)$$

At maximum velocity: $dv/dt = 0$

$$\therefore \frac{dv}{dt} = -a\omega^2 \cos \omega t - 4a\omega^2 \cos 2\omega t = 0$$

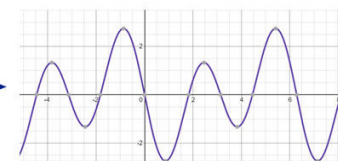
$$\Rightarrow 8 \cos^2 \omega t + \cos \omega t - 4 = 0$$

Rejecting -ve root, $\cos \omega t = 0.64, -0.77$

$$\Rightarrow \sin \omega t = \pm 0.77, \pm 0.64$$

From, (1) $\Rightarrow v = -a\omega (\sin \omega t + 2 \sin \omega t \cos \omega t)$

$$\Rightarrow v_{\max} = \pm 2.7 a\omega, \pm 1.3 a\omega$$



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4.12. The superposition of two harmonic oscillations of the same direction results in the oscillation of a point according to the law $x = a \cos 2.1t \cos 50.0t$, where t is expressed in seconds. Find the angular frequencies of the constituent oscillations and the period with which they beat. $\rightarrow$ i.e., time period of beat = $1/f_{\text{beat}}$

$$\begin{aligned} x &= \frac{a}{2} (2 \cos 2.1t \cos 50t) \\ &= \frac{a}{2} [(\cos(2.1t + 50t) + \cos(50t - 2.1t))] \\ &= \frac{a}{2} \cos \underbrace{52.1t}_{\omega_1} + \frac{a}{2} \cos \underbrace{47.9t}_{\omega_2} \end{aligned}$$

Beat frequency:

$$\begin{aligned} f_{\text{beat}} &= f_1 - f_2 \\ &= \frac{\omega_1}{2\pi} - \frac{\omega_2}{2\pi} = \frac{\omega_1 - \omega_2}{2\pi} \end{aligned}$$

$\therefore$ T of beats:

$$T_{\text{beat}} = \frac{1}{f_{\text{beat}}} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{2\pi}{(52.1 - 47.9)} = \frac{\pi}{2.1} = 1.5 \text{ sec}$$

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4.13. A point A oscillates according to a certain harmonic law in the reference frame K' which in its turn performs harmonic oscillations relative to the reference frame K . Both oscillations occur along the same direction. When the K' frame oscillates at the frequency 20 or 24 Hz, the beat frequency of the point A in the K frame turns out to be equal to ν . At what frequency of oscillation of the frame K' will the beat frequency of the point A become equal to 2ν ?

$$x_{AK} = x_{AK'} + x_{K'K}$$

Beats ν f_2 f_1

2 superimposed SHM's with frequencies f_1 and f_2

Case 1

$$f_1 \begin{cases} 20 \\ 24 \end{cases}$$

$$\text{Beat frequency: } \nu = 24 - f_2 = f_2 - 20 = \nu$$

$$\Rightarrow f_2 = 22, \nu = 2$$

$$f_2$$

Case 2

$$f_1 = ?$$

$$f_2 = 22$$

$$\text{If beat frequency} = 2\nu = 4$$

$$\text{either } f_1 - 22 = 4 \quad \text{or } 22 - f_1 = 4$$

$$\Rightarrow f_1 = 26 \quad \text{or } f_1 = 18$$

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4.14. A point moves in the plane xy according to the law $x = a \sin \omega t$, $y = b \cos \omega t$, where a , b , and ω are positive constants.

Find:

- (a) the trajectory equation $y(x)$ of the point and the direction of its motion along this trajectory;
 (b) the acceleration $\vec{a}$ of the point as a function of its radius vector $\vec{r}$ relative to the origin of coordinates.

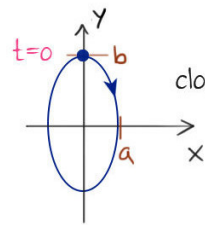
A $x = a \sin \omega t$
 $y = b \cos \omega t$

double
differentiate $\rightarrow$

B $a_x = -\omega^2 x$
 $a_y = -\omega^2 y$

Eliminating t

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{elliptical})$$



clockwise, starting from top.

In vector form:

$$\begin{aligned} \vec{a} &= \vec{a}_x + \vec{a}_y = -\omega^2 x \hat{i} - \omega^2 y \hat{j} \\ &= -\omega^2 (x \hat{i} + y \hat{j}) \\ &= -\omega^2 \vec{r} \end{aligned}$$

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4.15. Find the trajectory equation $y(x)$ of a point if it moves according to the following laws:

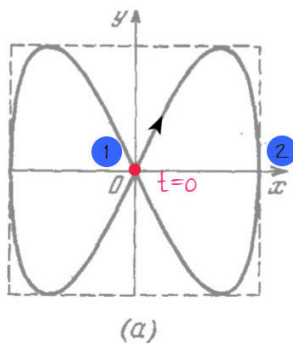
(a) $x = a \sin \omega t$, $y = a \sin 2\omega t$;

(b) $x = a \sin \omega t$, $y = a \cos 2\omega t$.

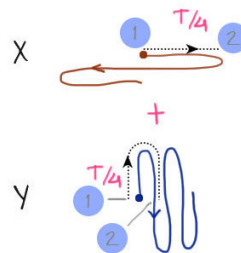
Plot these trajectories.

A $y = 2a \sin \omega t \cos \omega t$

$$\Rightarrow y = 2a \left(\frac{x}{a} \right) \sqrt{1 - \frac{x^2}{a^2}}$$

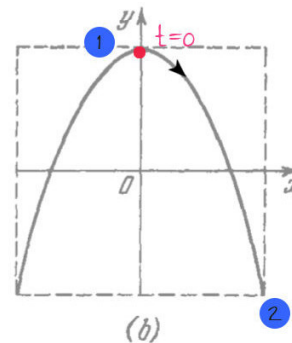


In, $T = \frac{2\pi}{\omega}$ time

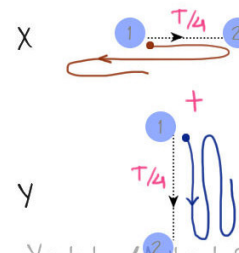


B $y = a(1 - 2\sin^2 \omega t)$

$$\Rightarrow y = a \left(1 - 2 \frac{x^2}{a^2} \right) \quad \text{Inverted parabola}$$



In, $T = \frac{2\pi}{\omega}$ time



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4.16. A particle of mass m is located in a unidimensional potential field where the potential energy of the particle depends on the coordinate x as $U(x) = U_0(1 - \cos ax)$; U_0 and a are constants. Find the period of small oscillations that the particle performs about the equilibrium position.

$$F = -\frac{dU}{dx} = -aU_0 \sin ax \approx -\underbrace{(a^2 U_0)}_k x \quad (\text{For small } x, \sin ax = ax)$$

$$\Rightarrow T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{a^2 U_0}} //$$

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In a perfect SHM

$$F = -k(x - x_0)$$

$$\Rightarrow U = \int -F \cdot dx$$

$$= \int k(x - x_0) dx$$

$$= \frac{kx^2}{2} - kx_0 x \quad (1) //$$

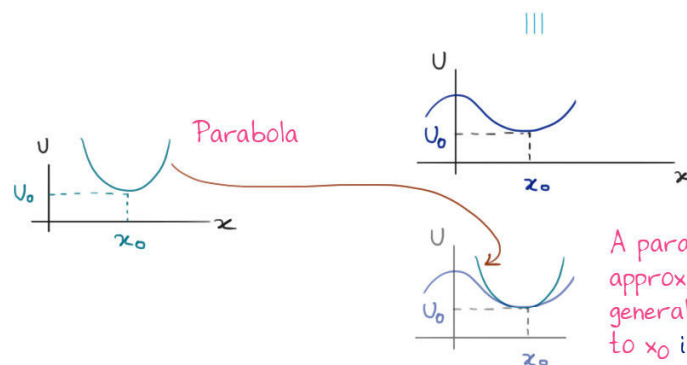
$$\Rightarrow U' = k(x - x_0) \quad (2)$$

$$\Rightarrow U'' = k \text{ (constant)} \quad (3)$$

∴ Time period for a U , for small displacement about its mean position can be approximated to that of perfect SHM:

$$\text{i.e. } T \approx 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{U''_{x_0}}} //$$

For a general U_x with minima at x_0 :



A parabolic curve approximately fits the general curve very close to x_0 if

From

- 1 $U_{x_0} = U_{x_0} = U_0 \quad \checkmark$
- 2 $U'_{x_0} = U'_{x_0} = 0 \quad \checkmark$
- 3 $U''_{x_0} = U''_{x_0} = k \quad \checkmark$

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4.17. Solve the foregoing problem if the potential energy has the form $U(x) = a/x^2 - b/x$, where a and b are positive constants.

$$F = -\frac{dU}{dx} = \frac{2a}{x^3} - \frac{b}{x^2}$$

Let mean position be at x_0

$$\Rightarrow \frac{2a}{x_0^3} - \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{2a}{b}$$

$$U = \frac{a}{x^2} - \frac{b}{x}$$

$$U' = -F$$

$$U'' = -\frac{dF}{dx} = -\left(-\frac{6a}{x^4} + \frac{2b}{x^3}\right)$$

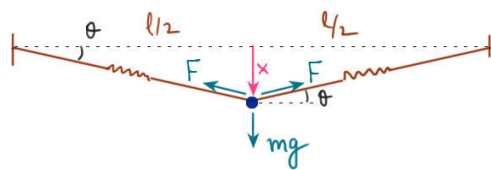
We know,
 $T \approx 2\pi \sqrt{\frac{m}{U''(x_0)}}$

$x = 2a/b$ $\rightarrow$ $U''(x_0)$

$$= 2\pi \sqrt{\frac{m}{(b^4/8a^3)}} = 2\pi \sqrt{\frac{8a^3m}{b^4}}$$

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4.18. Find the period of small oscillations in a vertical plane performed by a ball of mass $m = 40$ g fixed at the middle of a horizontally stretched string $l = 1.0$ m in length. The tension of the string is assumed to be constant and equal to $F = 10$ N.



When ball is down by x :

$$F_{\text{down}} = mg - 2F \sin \theta$$

$$\Rightarrow ma \approx mg - 2F \frac{x}{l/2} \quad (\sin \theta \approx \tan \theta = \frac{x}{l/2})$$

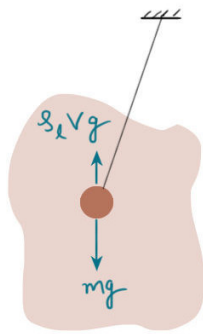
$$\Rightarrow m \frac{d^2x}{dt^2} - \frac{4F}{ml} \left(x - \frac{mgl}{4F} \right) = 0$$

Equation of SHM: ω^2 mean position

$$\therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{ml}{4F}}$$

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4.19. Determine the period of small oscillations of a mathematical pendulum, that is a ball suspended by a thread $l = 20$ cm in length, if it is located in a liquid whose density is $\eta = 3.0$ times less than that of the ball. The resistance of the liquid is to be neglected.



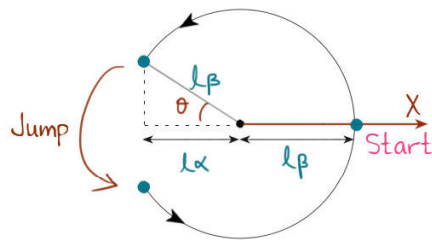
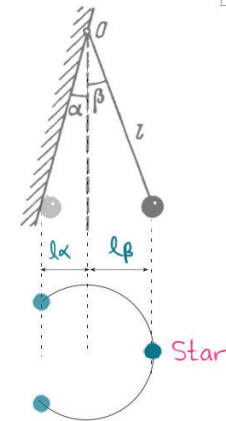
Effective gravity:

$$g_{\text{eff}} = \frac{mg - \rho_l V g}{m} = g \left(1 - \frac{\rho_l}{\rho_s}\right)$$

$$\therefore T = 2\pi \sqrt{\frac{l}{g_{\text{eff}}}} = 2\pi \sqrt{\frac{l}{g \left(1 - \frac{\rho_l}{\rho_s}\right)}} = 2\pi \sqrt{\frac{l}{g \left(1 - \frac{1}{\eta}\right)}} = 2\pi \sqrt{\frac{\eta l}{g(\eta - 1)}}$$

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4.20. A ball is suspended by a thread of length l at the point O on the wall, forming a small angle α with the vertical (Fig. 4.1). Then the thread with the ball was deviated through a small angle β ($\beta > \alpha$) and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.



$$T = \left(\frac{2\pi - 2\theta}{2\pi}\right) T_{\text{total}}$$

$$= \frac{(\pi - \theta)}{\pi} 2\pi \sqrt{\frac{l}{g}}$$

$$(\pi - \cos^{-1} \frac{\alpha}{\beta}) 2\sqrt{\frac{l}{g}}$$

$$\left(\frac{\pi}{2} + \alpha\right) 2\sqrt{\frac{l}{g}} \quad (\because \pi - \theta = \frac{\pi}{2} + \alpha)$$

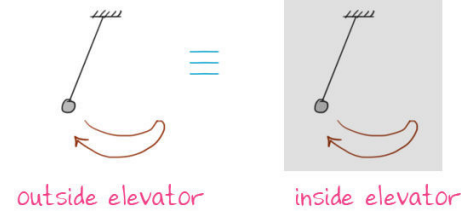
$$= \left(\frac{\pi}{2} + \sin^{-1} \frac{\alpha}{\beta}\right) 2\sqrt{\frac{l}{g}}$$

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4.21. A pendulum clock is mounted in an elevator car which starts going up with a constant acceleration w , with $w < g$. At a height h the acceleration of the car reverses, its magnitude remaining constant. How soon after the start of the motion will the clock show the right time again?

To eventually show the correct time: Pendulum, must swing equal number of times normally and inside the elevator

$$\therefore \frac{t_{\text{total}}}{T_{\text{normal}}} = \frac{t_{\text{up}}}{T_{\text{up}}} + \frac{t_{\text{down}}}{T_{\text{down}}}$$

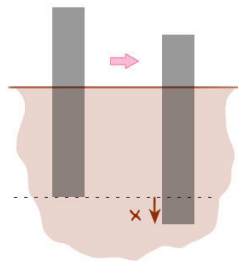


$$\Rightarrow \frac{t_t}{2\pi\sqrt{\frac{l}{g}}} = \frac{\sqrt{\frac{2h}{w}}}{2\pi\sqrt{\frac{l}{g+w}}} + \frac{t_t - \sqrt{\frac{2h}{g}}}{2\pi\sqrt{\frac{l}{g-w}}} \Rightarrow t_t = \sqrt{\frac{2h}{w}} \left(\frac{\sqrt{1+\beta} - \sqrt{1-\beta}}{1 - \sqrt{1-\beta}} \right)$$

where, $\beta = \frac{w}{g}$

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4.22. Calculate the period of small oscillations of a hydrometer (Fig. 4.2) which was slightly pushed down in the vertical direction. The mass of the hydrometer is $m = 50$ g, the radius of its tube is $r = 3.2$ mm, the density of the liquid is $\rho = 1.00$ g/cm³. The resistance of the liquid is assumed to be negligible.



Initially its at rest

$$\therefore \rho_l V g - mg = 0$$

After pushing down, restoring force

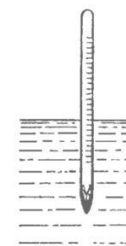
$$F_{\text{up}} = \rho_l (\cancel{V} + \Delta V) g - \cancel{mg}$$

$$= \rho_l \Delta V g$$

$$= \rho_l (\pi r^2 x) g$$

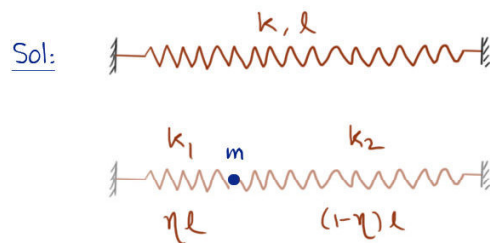
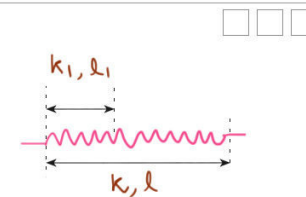
$$= \underbrace{(\rho_l \pi r^2 g)}_k x$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\rho_l \pi r^2 g}} = \sqrt{\frac{4\pi m}{\rho_l r^2 g}}$$



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4.23. A non-deformed spring whose ends are fixed has a stiffness $k = 13 \text{ N/m}$. A small body of mass $m = 25 \text{ g}$ is attached at the point removed from one of the ends by $\eta = 1/3$ of the spring's length. Neglecting the mass of the spring, find the period of small longitudinal oscillations of the body. The force of gravity is assumed to be absent.



We know,
For any part of a spring:

$$k_1 l_1 = k l = \text{constant}$$

$$\Rightarrow k_1 = \left(\frac{l}{l_1}\right) k$$

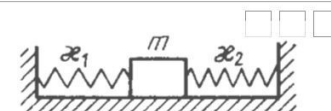
Springs will act in parallel

$$\therefore T = 2\pi \sqrt{\frac{m}{k_{eq}}} = 2\pi \sqrt{\frac{m}{k_1 + k_2}} = 2\pi \sqrt{\frac{m \eta (1-\eta)}{k}}$$

$\swarrow$ $k \frac{l}{\eta l}$ $\searrow$ $k \frac{l}{(1-\eta)l}$

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4.24. Determine the period of small longitudinal oscillations of a body with mass m in the system shown in Fig. 4.3. The stiffness values of the springs are k_1 and k_2 . The friction and the masses of the springs are negligible.



Just like previous problem 4.23,
both springs are in parallel.

$$\therefore T = 2\pi \sqrt{\frac{m}{k_1 + k_2}}$$

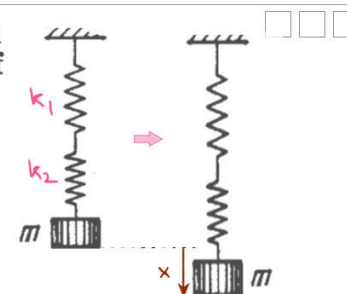
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4.25. Find the period of small vertical oscillations of a body with mass m in the system illustrated in Fig. 4.4. The stiffness values of the springs are k_1 and k_2 , their masses are negligible.

Constant force (mg), doesn't change Time period, it only shifts mean position.

System is similar to two springs in series in absence of gravity.

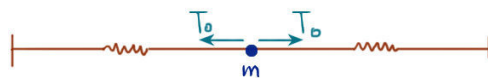
$$\therefore T = 2\pi \sqrt{\frac{m}{k_{eq}}} = 2\pi \sqrt{\frac{m(k_1+k_2)}{k_1 k_2}} //$$



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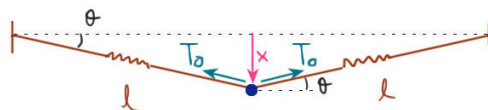
4.26. A small body of mass m is fixed to the middle of a stretched string of length $2l$. In the equilibrium position the string tension is equal to T_0 . Find the angular frequency of small oscillations of the body in the transverse direction. The mass of the string is negligible, the gravitational field is absent.

T_0 can be assumed constant



Restoring Force $\approx 2T_0 \sin \theta$

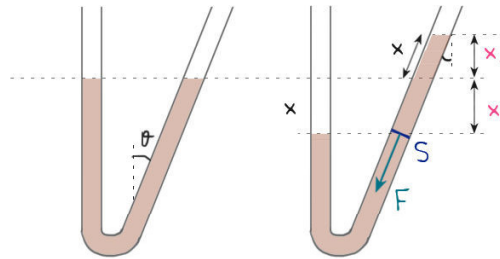
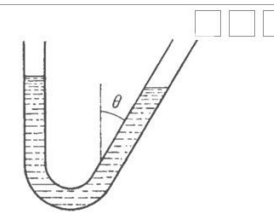
$$\approx 2T_0 \frac{x}{l} = \underbrace{\left(\frac{2T_0}{l}\right)}_k x$$



$$\Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2T_0}{ml}} //$$

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4.27. Determine the period of oscillations of mercury of mass $m = 200 \text{ g}$ poured into a bent tube (Fig. 4.5) whose right arm forms an angle $\theta = 30^\circ$ with the vertical. The cross-sectional area of the tube is $S = 0.50 \text{ cm}^2$. The viscosity of mercury is to be neglected.



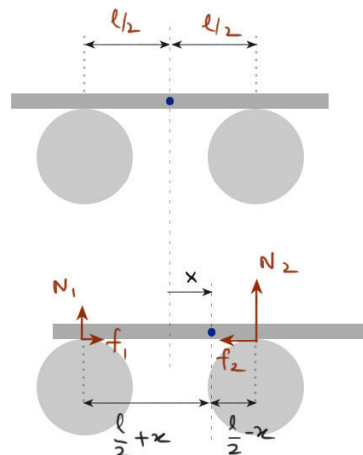
Restoring force

$$\begin{aligned} F &= P \cdot A \\ &= \rho g (x + x_1) S \\ &= \rho g x (1 + \cos \theta) S \\ &= \underbrace{\rho g S (1 + \cos \theta)}_k x \end{aligned}$$

$$\therefore T = 2\pi \sqrt{\frac{m}{\rho g S (1 + \cos \theta)}}$$

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4.28. A uniform rod is placed on two spinning wheels as shown in Fig. 4.6. The axes of the wheels are separated by a distance $l = 20 \text{ cm}$, the coefficient of friction between the rod and the wheels is $k = 0.18$. Demonstrate that in this case the rod performs harmonic oscillations. Find the period of these oscillations.



$$\begin{aligned} \text{Restoring force} &= f_2 - f_1 \\ &= k(N_2 - N_1) \\ &= \frac{2kmg}{l} x \end{aligned}$$

$$F: N_1 + N_2 = mg \quad (1)$$

$$\tau: N_2 \left(\frac{l}{2} - x \right) = N_1 \left(\frac{l}{2} + x \right) \quad (2)$$

$$\Rightarrow (N_2 - N_1) \frac{l}{2} = (N_1 + N_2) x$$

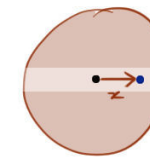
$$\Rightarrow N_2 - N_1 = \frac{2mgx}{l}$$

$$\therefore T = 2\pi \sqrt{\frac{l}{2kg}}$$

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4.29. Imagine a shaft going all the way through the Earth from pole to pole along its rotation axis. Assuming the Earth to be a homogeneous ball and neglecting the air drag, find:

- the equation of motion of a body falling down into the shaft;
- how long does it take the body to reach the other end of the shaft;
- the velocity of the body at the Earth's centre.



We know,

$$g_x = \frac{GMx}{R^3}$$

$$F_x = mg_x = \frac{GMm}{R^3} x = \frac{mg}{R} x$$

∴ Equation of motion:

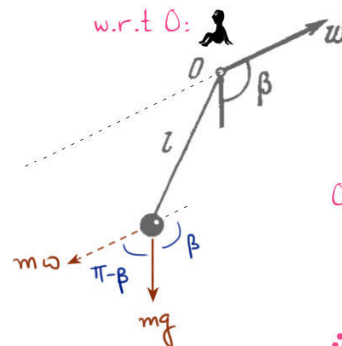
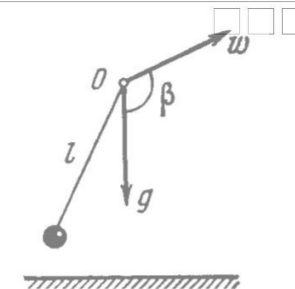
$$\begin{aligned} \text{A } x &= A \cos \omega t \\ &\quad \downarrow R \quad \downarrow \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{R}} = \sqrt{\frac{g}{R}} \\ &= R \cos \sqrt{\frac{g}{R}} t \end{aligned}$$

$$\text{B } v_0 = \omega A = \sqrt{\frac{g}{R}} \cdot R = \sqrt{gR}$$

$$\text{C } T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R}{g}}$$

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4.30. Find the period of small oscillations of a mathematical pendulum of length l if its point of suspension O moves relative to the Earth's surface in an arbitrary direction with a constant acceleration w (Fig. 4.7). Calculate that period if $l = 21$ cm, $w = g/2$, and the angle between the vectors w and g equals $\beta = 120^\circ$.



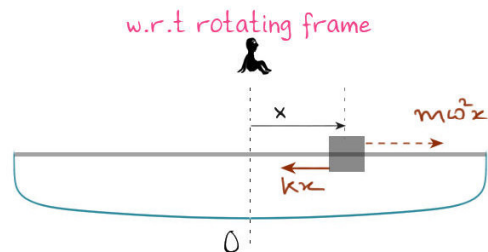
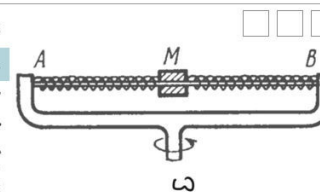
On the bob wrt O:

$$g_{\text{eff}} = \sqrt{w^2 + g^2 + 2wg \cos(\pi - \beta)}$$

$$\therefore T = 2\pi \sqrt{\frac{l}{g_{\text{eff}}}}$$

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4.31. In the arrangement shown in Fig. 4.8 the sleeve M of mass $m = 0.20$ kg is fixed between two identical springs whose combined stiffness is equal to $k = 20$ N/m. The sleeve can slide without friction over a horizontal bar AB . The arrangement rotates with a constant angular velocity $\omega = 4.4$ rad/s about a vertical axis passing through the middle of the bar. Find the period of small oscillations of the sleeve. At what values of ω will there be no oscillations of the sleeve?



$$\text{Restoring force} = kx - m\omega^2 x \\ = (k - m\omega^2) x$$

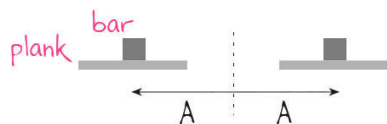
$$\therefore T = 2\pi \sqrt{\frac{m}{k - m\omega^2}}$$

$k > m\omega^2$ Stable equilibrium (it will come back to O)

No oscillations $\left\{ \begin{array}{l} k = m\omega^2 \text{ Neutral equilibrium (it will stay wherever it's kept)} \\ k < m\omega^2 \text{ Unstable equilibrium (it will go to outer end, if disturbed)} \end{array} \right.$

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4.32. A plank with a bar placed on it performs horizontal harmonic oscillations with amplitude $A = 10$ cm. Find the coefficient of friction between the bar and the plank if the former starts sliding along the plank when the Time period of oscillation of the plank becomes less than $T = 1.0$ s.



Bar slips when period of plank is less than T , $\mu = ?$

Sol: Bar will experience maximum force at extreme SHM position.

$$\therefore F_{\max} = m\omega^2 A$$

when it just slips $\left(\right.$

$$\mu mg = m\omega^2 A$$

$$\Rightarrow \mu = \frac{\omega^2 A}{g} = \left(\frac{2\pi}{T} \right)^2 \frac{A}{g} = \frac{4\pi^2 A}{T^2 g}$$

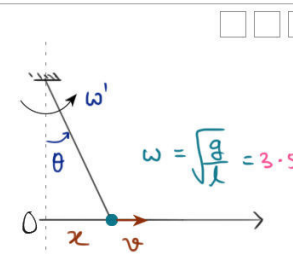
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4.33. Find the time dependence of the angle of deviation of a mathematical pendulum 80 cm in length if at the initial moment the pendulum

(a) was deviated through the angle 3.0° and then set free without push;

(b) was in the equilibrium position and its lower end was imparted the horizontal velocity 0.22 m/s;

(c) was deviated through the angle 3.0° and its lower end was imparted the velocity 0.22 m/s directed toward the equilibrium position.



A $x = A \cos \omega t$

$\theta = \theta_0 \cos \omega t$

B $v_0 = \omega A$

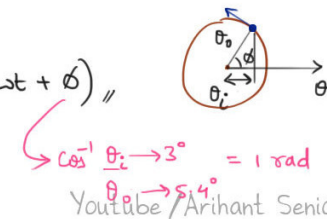
$\omega_0 = \omega \theta_0$
 $\Rightarrow \frac{v_0}{l} = \omega \theta_0$
 $\Rightarrow \theta_0 = \frac{v_0}{l \omega} = 4.5^\circ$

$\theta = \theta_0 \sin \omega t$

C $v = \omega \sqrt{A^2 - x^2}$

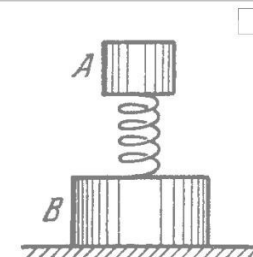
$\omega' = \omega \sqrt{\theta_0^2 - \theta^2}$
 $\Rightarrow \theta_0 = 5.4^\circ$

$\theta = \theta_0 \cos(\omega t + \phi)$



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4.34. A body A of mass $m_1 = 1.00$ kg and a body B of mass $m_2 = 4.10$ kg are interconnected by a spring as shown in Fig. 4.9. The body A performs free vertical harmonic oscillations with amplitude $a = 1.6$ cm and frequency $\omega = 25$ s⁻¹. Neglecting the mass of the spring, find the maximum and minimum values of force that this system exerts on the bearing surface.

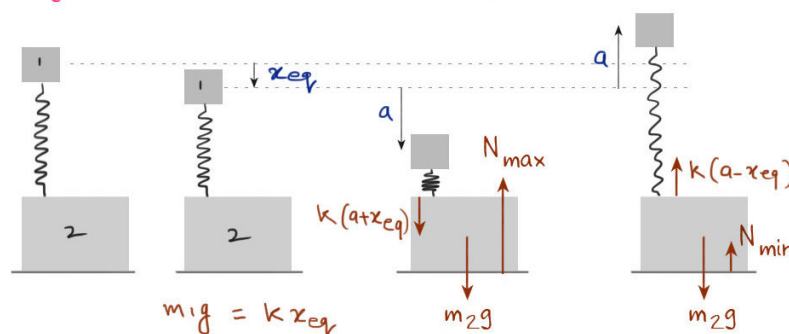


Natural length

At equilibrium

Max compression

Max expansion



Force balance on m_2 :

$N_{\max} = m_2 g + k(a + x_{eq})$
 $= (m_1 + m_2)g + k a$
 $= (m_1 + m_2)g + m_1 \omega^2 a$

$N_{\min} = m_2 g - k(a - x_{eq})$
 $= (m_1 + m_2)g - m_1 \omega^2 a$

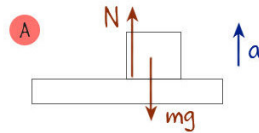
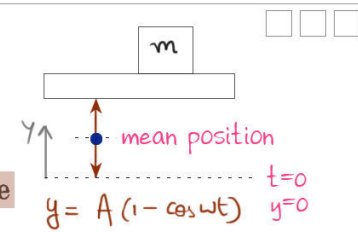
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4.35. A plank with a body of mass m placed on it starts moving straight up according to the law $y = A(1 - \cos \omega t)$, where y is the displacement from the initial position, $\omega = 11 \text{ s}^{-1}$. Find:

(a) the time dependence of the force that the body exerts on the plank if $A = 4.0 \text{ cm}$; plot this dependence;

(b) the minimum amplitude of oscillation of the plank at which the body starts falling behind the plank; in other words, $A_{\max}$, at which body doesn't fall behind.

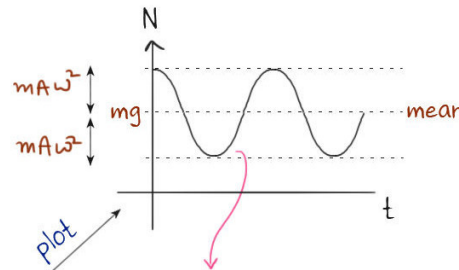
(c) the amplitude of oscillation of the plank at which the body springs up to a height $h = 50 \text{ cm}$ relative to the initial position (at the moment $t = 0$).



$$N - mg = ma$$

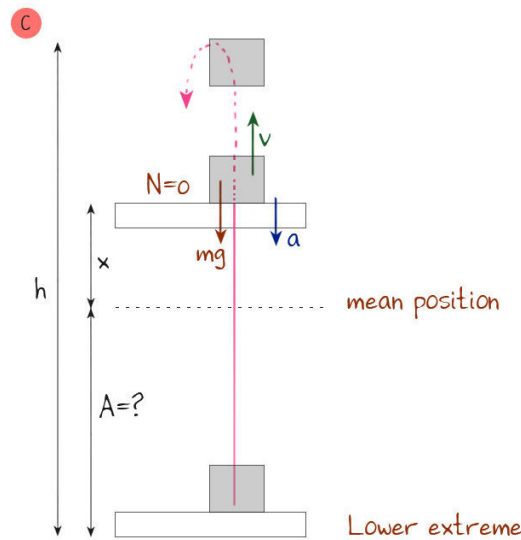
$$\Rightarrow N = m(g + a)$$

$$\Rightarrow N = m(g + A\omega^2 \cos \omega t)$$



B If graph touches X-axis, contact will break.
i.e. $mg = m A \omega^2 \Rightarrow A = g/\omega^2$

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Lets say block breaks contact at x , when $N=0$.

When $N=0$, only force is mg , so acceleration $a = \omega^2 x = g$ 1
After that moment, as x increases, lower plank will be pulled down harder, and upper block will break contact and free fall.

Blocks velocity at x : $v = \omega \sqrt{A^2 - x^2}$ 2

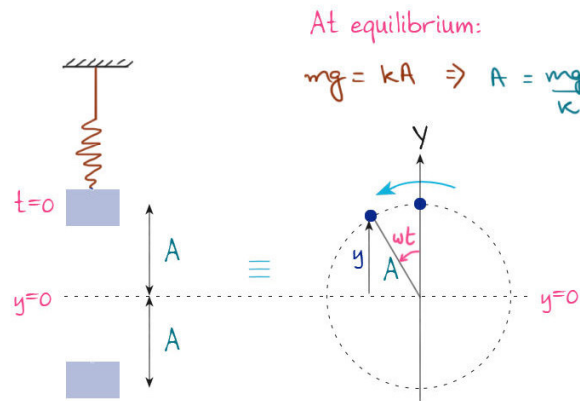
Height gained after breaking contact is $h - (A + x)$. So:

$$v^2 = 2g[h - (A + x)]$$
 3

From 1,2,3: $A = \frac{g}{\omega^2} \left(\omega \sqrt{\frac{2h}{g}} - 1 \right)$

4.36. A body of mass m was suspended by a non-stretched spring, and then set free without push. The stiffness of the spring is k . Neglecting the mass of the spring, find:

- (a) the law of motion $y(t)$, where y is the displacement of the body from the equilibrium position;
 (b) the maximum and minimum tensions of the spring in the process of motion.



At equilibrium:

$$mg = kA \Rightarrow A = \frac{mg}{k}$$

B At top, $T_{\min} = 0$

At bottom, $T_{\max} = k(2A)$

$$= k(2 \frac{mg}{k})$$

$$= 2mg \text{ (independent of } k)$$

A Clearly from phase diagram:

$$y = A \cos \omega t \quad // \quad \text{(Answer in book is for } y \text{ as displacement from unstretched end)}$$

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4.37. A particle of mass m moves due to the force $\vec{F} = -\alpha m \vec{r}$, where α is a positive constant, $\vec{r}$ is the radius vector of the particle relative to the origin of coordinates. Find the trajectory of its motion if at the initial moment $\vec{r} = r_0 \hat{i}$ and the velocity $\vec{v} = v_0 \hat{j}$, where $\hat{i}$ and $\hat{j}$ are the unit vectors of the x and y axes.

Lets break the motion in X and Y directions.

$$\begin{aligned} \vec{F} &= -\alpha m \vec{r} \\ &= -\alpha m (x \hat{i} + y \hat{j}) \end{aligned}$$

$$\Rightarrow F_x = -\alpha m x$$

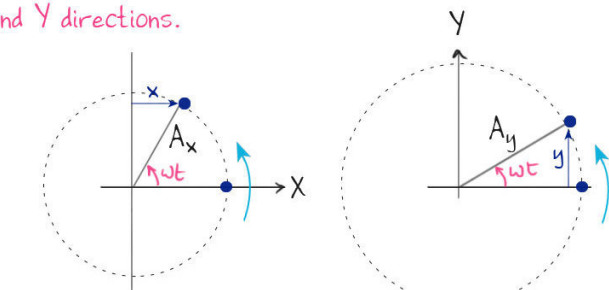
$$F_y = -\alpha m y$$

SHM in both X and Y directions.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\alpha m}{m}} = \sqrt{\alpha}$$

$$A_x = r_0$$

$$A_y = \frac{v_{\max}}{\omega} = \frac{v_0}{\sqrt{\alpha}}$$

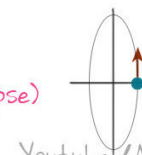
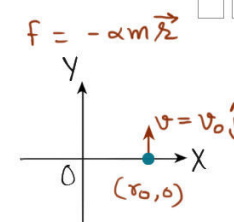


$$\therefore x = A_x \cos \omega t \quad (1)$$

$$\therefore y = A_y \sin \omega t \quad (2)$$

From 1,2 eliminating t :

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1 \Rightarrow \frac{x^2}{r_0^2} + \frac{y^2}{v_0^2 \alpha} = 1 \quad // \quad \text{(ellipse)}$$



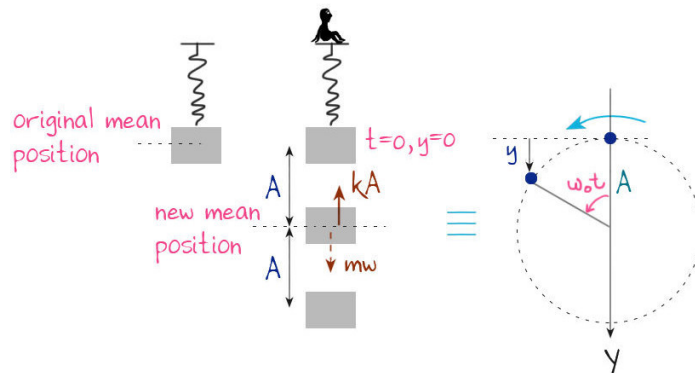
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4.38. A body of mass m is suspended from a spring fixed to the ceiling of an elevator car. The stiffness of the spring is k . At the moment $t = 0$ the car starts going up with an acceleration w . Neglecting the mass of the spring, find the law of motion $y(t)$ of the body relative to the elevator car if $y(0) = 0$ and $\dot{y}(0) = 0$. Consider the following two cases:

- (a) $w = \text{const}$;
 (b) $w = \alpha t$, where α is a constant.

Method 1 A

w.r.t car



Forces due to original extension and mg are already balanced, so we didn't show them in equation 1 as $k(A+x_0) = mw + mg$

∴ Situation is similar to system kept on horizontal table which starts to accelerate.

$$\therefore kA = mw \quad (1)$$

$$\Rightarrow A = \frac{mw}{k}$$

$$\text{Also, } \omega_0 = \sqrt{\frac{k}{m}}$$

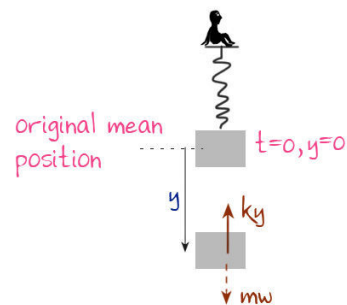
From phase diagram:

$$y = A(1 - \cos \omega_0 t) \\ = \frac{mw}{k}(1 - \cos \omega_0 t) //$$

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A Method 2

w.r.t car



w.r.t car, at general y :

$$m\ddot{y} = mw - ky + mg - ky_0$$

$$\Rightarrow \frac{d^2 y}{dt^2} + \underbrace{\frac{k}{m}}_{\omega_0^2} \left(y - \frac{mw}{k} \right) = 0$$

At new mean:

$$kA = mw$$

$$\Rightarrow A = \frac{mw}{k}$$

∴ Solution: $y - \frac{mw}{k} = A \sin(\omega_0 t + \phi) \quad (1)$ $\Rightarrow y = \frac{mw}{k}(1 - \cos \omega_0 t) //$

B Equation 1 still holds with $w = \alpha t$

$$\left[\therefore \frac{d^2 y}{dt^2} = \frac{d^2}{dt^2} \left(y - \frac{m\alpha t}{k} \right) \right] \quad (0 \text{ at } t=0, y=0)$$

$$\text{Solution: } y - \frac{m\alpha t}{k} = A \sin(\omega_0 t + \phi)$$

differentiate $\downarrow$ $(-\frac{1}{\omega_0^2} \frac{m\alpha}{k} \text{ at } t=0, \dot{y}=0)$

$$\dot{y} - \frac{m\alpha}{k} = A\omega_0 \cos \omega_0 t$$

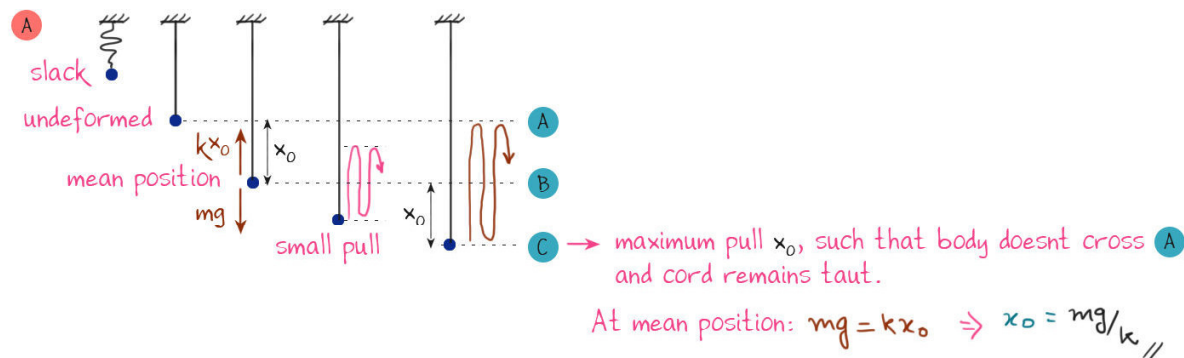
$$\Rightarrow y - \frac{m\alpha t}{k} = -\frac{m\alpha}{k\omega_0^2} \sin \omega_0 t$$

$$k/m \rightarrow \omega_0^2$$

$$\Rightarrow y = \frac{\alpha}{\omega_0^3} (\omega_0 t - \sin \omega_0 t) //$$

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4.39. A body of mass $m = 0.50 \text{ kg}$ is suspended from a rubber cord with elasticity coefficient $k = 50 \text{ N/m}$. Find the maximum distance over which the body can be pulled down for the body's oscillations to remain harmonic. What is the energy of oscillation in this case?



B M1 Energy of oscillation is KE at mean position

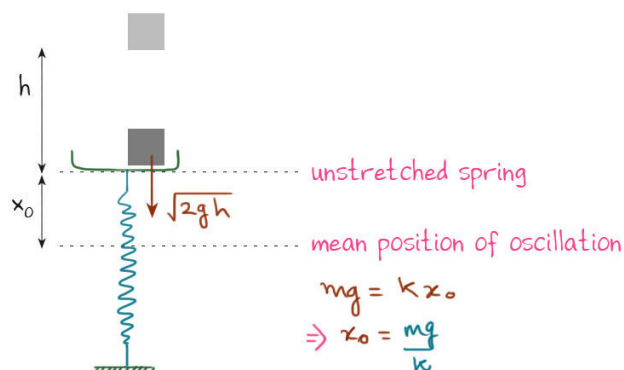
$$E = \frac{1}{2} m v_{\max}^2 = \frac{1}{2} m (\omega A)^2 = \frac{1}{2} m \cdot \frac{k}{m} \cdot \frac{m^2 g^2}{k^2} = \frac{1}{2} \frac{m^2 g^2}{k} //$$

M2 PE at extreme position

$$E = \frac{1}{2} k A^2 = \frac{1}{2} k \left(\frac{mg}{k} \right)^2 = \frac{m^2 g^2}{2k} //$$

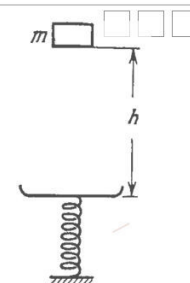
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4.40. A body of mass m fell from a height h onto the pan of a spring balance (Fig. 4.10). The masses of the pan and the spring are negligible, the stiffness of the latter is k . Having stuck to the pan, the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and the energy of these oscillations.



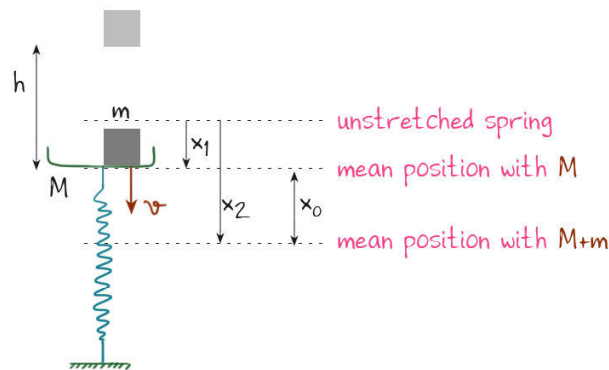
A $v = \omega \sqrt{A^2 - x_0^2}$
 $\Rightarrow \sqrt{2gh} = \sqrt{\frac{k}{m}} \sqrt{A^2 - \frac{m^2 g^2}{k^2}}$
 $\Rightarrow A = \frac{mg}{k} \sqrt{1 + \frac{2kh}{mg}}$

B $E_{\text{osc}} = \frac{1}{2} k A^2 //$



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4.41. Solve the foregoing problem for the case of the pan having a mass M . Find the oscillation amplitude in this case.



$$p_i = p_f$$

$$m\sqrt{2gh} = (m+M)v$$

$$\Rightarrow v = \frac{m\sqrt{2gh}}{m+M}$$

$$Mg = kx_1$$

$$(M+m)g = kx_2$$

$$\therefore x_0 = x_2 - x_1 = \frac{mg}{k}$$

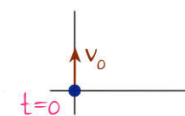
$$v = \omega \sqrt{A^2 - x_0^2}$$

$$\Rightarrow \frac{m\sqrt{2gh}}{(m+M)} = \sqrt{\frac{k}{m+M}} \sqrt{A^2 - \frac{m^2 g^2}{k^2}}$$

$$\Rightarrow A = \frac{mg}{k} \sqrt{1 + \frac{2kh}{g(m+M)}} //$$

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4.42. A particle of mass m moves in the plane xy due to the force varying with velocity as $\mathbf{F} = a(\dot{y}\mathbf{i} - \dot{x}\mathbf{j})$, where a is a positive constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. At the initial moment $t = 0$ the particle was located at the point $x = y = 0$ and possessed a velocity $\mathbf{v}_0$ directed along the unit vector $\mathbf{j}$. Find the law of motion $x(t)$, $y(t)$ of the particle, and also the equation of its trajectory.



$$F_x = m\ddot{x} = a\dot{y}$$

Integrate

$$m\dot{x} = ay + C_1$$

($\because$ at $y=0, \dot{x}=0$)

$$\Rightarrow \dot{x} = \frac{a}{m}y$$

$$\Rightarrow \ddot{y} = -\frac{a}{m}\left(\frac{a}{m}y\right)$$

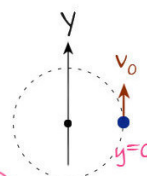
$$\Rightarrow \ddot{y} = -\left(\frac{a^2}{m^2}\right)y$$

SHM

$$\Rightarrow y = A \sin \omega t$$

$\omega A = v_0$

$$\Rightarrow y = \frac{v_0}{\omega} \sin \omega t //$$



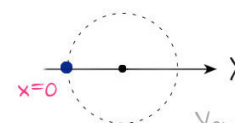
Putting 2 in 1:

$$\dot{x} = \left(\frac{a}{m}\right) \frac{v_0}{\omega} \sin \omega t$$

Integrate

$$x = -\frac{v_0}{\omega} \cos \omega t + C$$

$$\Rightarrow x = \frac{v_0}{\omega} (1 - \cos \omega t) //$$



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Trajectory:

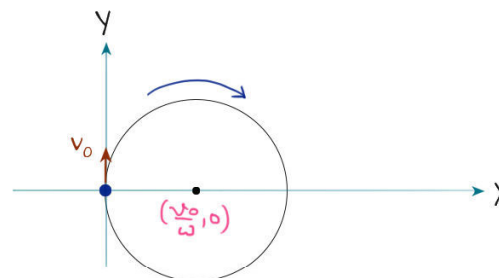
$$y = \frac{v_0}{\omega} \sin \omega t \quad x = \frac{v_0}{\omega} (1 - \cos \omega t)$$

Eliminating t : $\left(x - \frac{v_0}{\omega}\right)^2 + y^2 = \frac{v_0^2}{\omega^2}$

∴ It will travel in a circle

Radius: v_0/ω

Velocity: $\sqrt{\dot{x}^2 + \dot{y}^2} = v_0$ (constant)

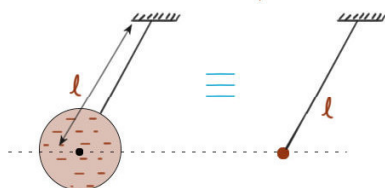


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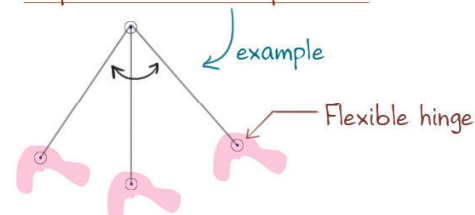
4.43. A pendulum is constructed as a light thin-walled sphere of radius R filled up with water and suspended at the point O from a light rigid rod (Fig. 4.11). The distance between the point O and the centre of the sphere is equal to l . How many times will the small oscillations of such a pendulum change after the water freezes? The viscosity of water and the change of its volume on freezing are to be neglected.



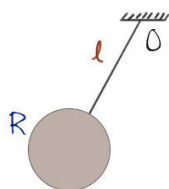
A When water is in liquid state, system is approximated as simple (mathematical) pendulum:



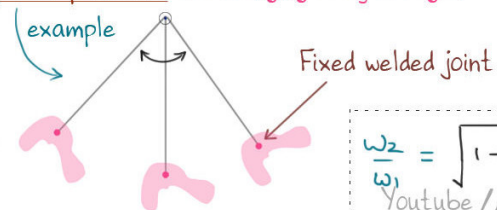
$$\omega_1 = \sqrt{\frac{g}{l}}$$



B When water is frozen, system acts as compound (physical) pendulum (as swinging body is rigid)



$$\omega_2 = \sqrt{\frac{mgl_{cm}}{I_0}} = \sqrt{\frac{mgl}{ml^2 + \frac{2}{5}mR^2}}$$



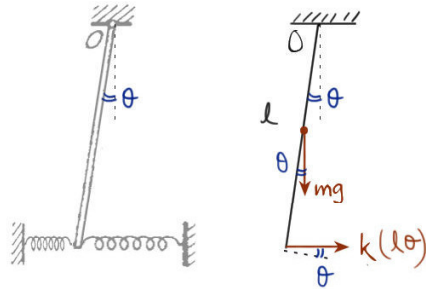
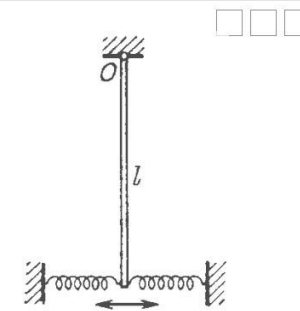
$$\frac{\omega_2}{\omega_1} = \sqrt{1 + \frac{2}{5} \left(\frac{R}{l}\right)^2}$$

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4.44. Find the frequency of small oscillations of a thin uniform vertical rod of mass m and length l hinged at the point O (Fig. 4.12). The combined stiffness of the springs is equal to k . The mass of the springs is negligible.

Restoring torque:

$$\begin{aligned}\tau_0 &= mg \sin \theta \frac{l}{2} + k l \theta \cos \theta \cdot l \\ &= \underbrace{\left(mg \frac{l}{2} + k l^2 \right)}_C \theta\end{aligned}$$



We know, if: $\tau_0 = -C\theta$
 $\Rightarrow I_0 \alpha = -C\theta$
 $\Rightarrow \alpha = -\left(\frac{C}{I_0}\right)\theta$
 ω^2

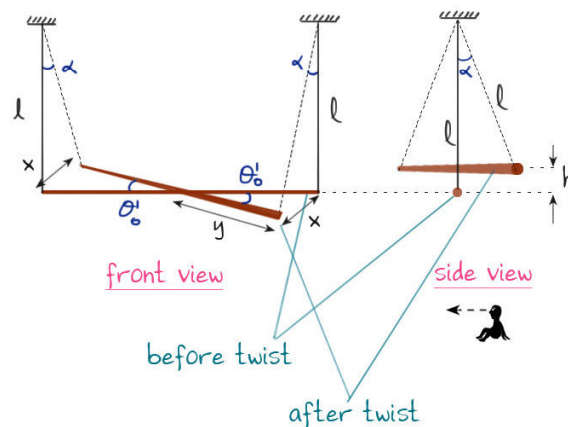
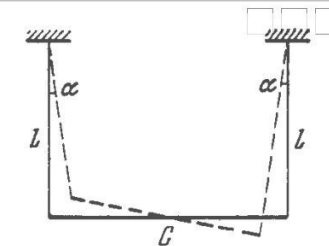
$$\therefore \omega = \sqrt{\frac{C}{I_0}} = \sqrt{\frac{\left(\frac{mg}{2} + kl\right)l}{\frac{ml^2}{3}}} = \sqrt{\frac{3g}{2l} \left(1 + \frac{2kl}{mg}\right)}$$

Youtube / Arihant Senior

4.45. A uniform rod of mass $m = 1.5$ kg suspended by two identical threads $l = 90$ cm in length (Fig. 4.13) was turned through a small angle about the vertical axis passing through its middle point C . The threads deviated in the process through an angle $\alpha = 5.0^\circ$. Then the rod was released to start performing small oscillations.

Find:

- the oscillation period;
- the rod's oscillation energy.



Let the rod length be $2y$.

B Energy of oscillation = ΔPE (i.e. $PE_{\text{extreme}} - PE_{\text{equilibrium}}$)

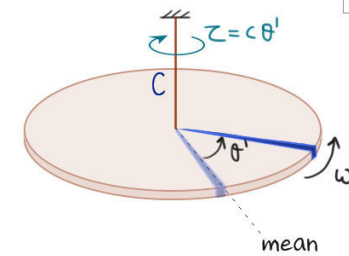
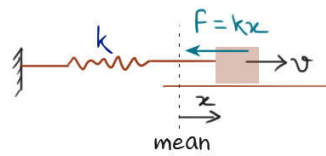
$$= mgh = mgl(1 - \cos \alpha) \approx mgl \frac{\alpha^2}{2}$$

A Also, energy of oscillation = KE at equilibrium

$$\begin{aligned}\Rightarrow mgl \frac{\alpha^2}{2} &= \frac{1}{2} I \omega_{\text{max}}^2 \\ \alpha l &= y \theta_0' \quad \frac{m(2y)^2}{12} \quad \omega = \omega \theta_0' \text{ (Similar to } v_{\text{max}} = \omega A)\end{aligned}$$

$$\Rightarrow mgl \frac{y^2 \theta_0'^2}{2l^2} = \frac{1}{2} \frac{m(4y^2)}{12} \cdot \omega^2 \theta_0'^2 \Rightarrow \omega = \sqrt{\frac{3g}{2l}}$$

Youtube / Arihant Senior



Similarities between Linear and Torsional SHM:

$$F = -kx$$

$$\Rightarrow a = -\left(\frac{k}{m}\right)x \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$$

$$v = \omega \sqrt{A^2 - x^2}$$

$$PE = -\int \vec{F}_{int} \cdot d\vec{x} = \int kx dx = \frac{1}{2}kx^2$$

$$E_{osc} = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 \rightarrow \omega A$$

$$\tau = -C\theta$$

$$\Rightarrow \alpha = -\left(\frac{C}{I}\right)\theta \Rightarrow T = 2\pi \sqrt{\frac{I}{C}}$$

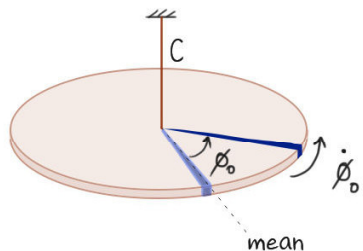
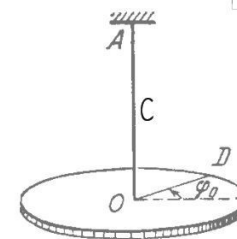
$$\omega' = \omega \sqrt{\theta_0^2 - \theta^2}$$

$$PE = -\int \vec{\tau}_{int} \cdot d\vec{\theta} = \int C\theta d\theta = \frac{1}{2}C\theta^2$$

$$E_{osc} = \frac{1}{2}C\theta_0^2 = \frac{1}{2}I\omega_0'^2 \rightarrow \omega\theta_0'$$

Youtube / Arihant Senior

4.46. An arrangement illustrated in Fig. 4.14 consists of a horizontal uniform disc D of mass m and radius R and a thin rod AO whose torsional coefficient is equal to C . Find the amplitude and the energy of small torsional oscillations if at the initial moment the disc was deviated through an angle ϕ_0 from the equilibrium position and then imparted an angular velocity $\dot{\phi}_0$.



$$\text{A } \omega' = \omega \sqrt{\theta_0^2 - \theta^2} \rightarrow \dot{\phi}_0 = \sqrt{\frac{C}{I}} \sqrt{\phi_{0max}^2 - \phi_0^2}$$

$$\Rightarrow \dot{\phi}_0 = \sqrt{\frac{C}{mR^2/2}} \sqrt{\phi_{0max}^2 - \phi_0^2}$$

$$\Rightarrow \phi_{0max} = \sqrt{\frac{mR^2}{2C} \dot{\phi}_0^2 + \phi_0^2}$$

$$\text{B } E_{osc} = \frac{1}{2}C\phi_{0max}^2$$

Youtube / Arihant Senior

4.47. A uniform rod of mass m and length l performs small oscillations about the horizontal axis passing through its upper end. Find the mean kinetic energy of the rod averaged over one oscillation period if at the initial moment it was deflected from the vertical by an angle θ_0 and then imparted an angular velocity $\dot{\theta}_0$.

Let the equation of torsional SHM be:

$$\theta' = \theta_0' \sin \omega t$$

$$\Rightarrow \omega' = \theta_0' \omega \cos \omega t$$

$$\Rightarrow KE = \frac{1}{2} I \omega'^2 = \frac{1}{2} I \theta_0'^2 \omega^2 \cos^2 \omega t$$

E_{osc}

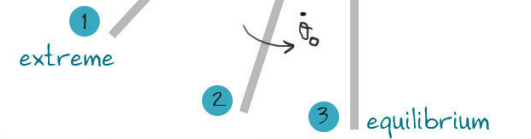
$$\Rightarrow \langle KE \rangle = \frac{\int_0^T KE dt}{T} = E_{osc} \left[\frac{\int_0^{2\pi/\omega} \cos^2 \omega t dt}{2\pi/\omega} \right]$$

$\frac{1}{2}$

$$\therefore \langle KE \rangle = \frac{E_{osc}}{2}$$

It applies to linear SHM too.

Sol:



$$E_{osc} = PE_{extreme} - PE_{equilibrium}$$

$$= PE_1 - PE_3$$

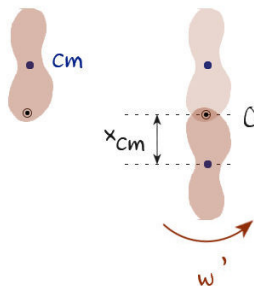
$$= (PE_1 - PE_2) + (PE_2 - PE_3)$$

$$= \frac{1}{2} I (\dot{\theta}_0)^2 + mg \frac{l}{2} (1 - \cos \theta) \approx \frac{1}{2} \theta^2$$

$$\therefore \langle KE \rangle = \frac{E_{osc}}{2} = \frac{1}{2} \left(\frac{ml^2}{6} \dot{\theta}_0^2 + \frac{mg l}{4} \theta^2 \right)$$

Youtube / Arihant Senior

4.48. A physical pendulum is positioned so that its centre of gravity is above the suspension point. From that position the pendulum started moving toward the stable equilibrium and passed it with an angular velocity ω' . Neglecting the friction find the period of small oscillations of the pendulum.



By Energy conservation:

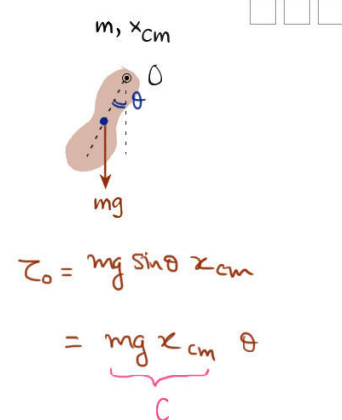
$$mg(2x_{cm}) = \frac{1}{2} I_0 \omega'^2$$

$$\Rightarrow \frac{I_0}{mg x_{cm}} = \frac{4}{\omega'^2}$$

In physical pendulum:

$$T = 2\pi \sqrt{\frac{I_0}{C}} = 2\pi \sqrt{\frac{I_0}{mg x_{cm}}} = 2\pi \sqrt{\frac{4}{\omega'^2}} = \frac{4\pi}{\omega'}$$

This is NOT oscillation frequency



$$\tau_0 = mg \sin \theta x_{cm}$$

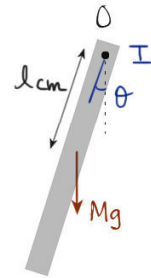
$$= mg x_{cm} \theta$$

C

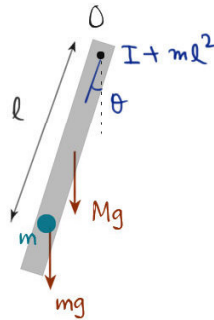
Youtube / Arihant Senior

4.49. A physical pendulum performs small oscillations about the horizontal axis with frequency $\omega_1 = 15.0 \text{ s}^{-1}$. When a small body of mass $m = 50 \text{ g}$ is fixed to the pendulum at a distance $l = 20 \text{ cm}$ below the axis, the oscillation frequency becomes equal to $\omega_2 = 10.0 \text{ s}^{-1}$. Find the moment of inertia of the pendulum relative to the oscillation axis.

Let moment of inertia by I .



$$\tau_0 = \underbrace{Mg l_{cm}}_{C_1} \theta$$



$$\tau_0 = \underbrace{(Mg l_{cm} + mgl)}_{C_2 = C_1 + mgl} \theta$$

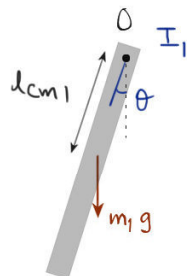
$$\omega_1 = \sqrt{\frac{C_1}{I}} \quad (1)$$

$$\omega_2 = \sqrt{\frac{C_2}{I_2}} = \sqrt{\frac{C_1 + mgl}{I + ml^2}} \quad (2)$$

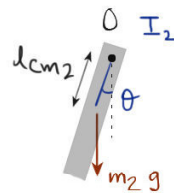
$$\text{From 1,2: } I = \frac{mgl - ml^2\omega_2^2}{\omega_2^2 - \omega_1^2} //$$

Youtube / Arihant Senior

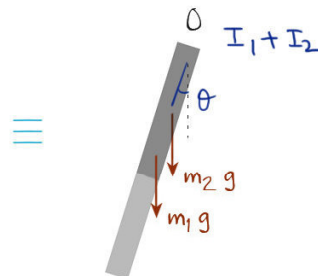
4.50. Two physical pendulums perform small oscillations about the same horizontal axis with frequencies ω_1 and ω_2 . Their moments of inertia relative to the given axis are equal to I_1 and I_2 respectively. In a state of stable equilibrium the pendulums were fastened rigidly together. What will be the frequency of small oscillations of the compound pendulum?



$$\tau_0 = \underbrace{m_1 g l_{cm1}}_{C_1} \theta$$



$$\tau_0 = \underbrace{m_2 g l_{cm2}}_{C_2} \theta$$



$$\tau_0 = \underbrace{(m_1 g l_{cm1} + m_2 g l_{cm2})}_{\therefore C = C_1 + C_2} \theta$$

Sol:

$$\omega_1 = \sqrt{\frac{C_1}{I_1}}$$

$$\omega_2 = \sqrt{\frac{C_2}{I_2}}$$

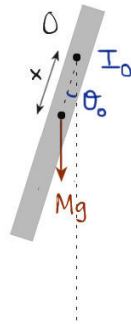
$$\omega_{eq} = \sqrt{\frac{C_1 + C_2}{I_1 + I_2}}$$

$$= \sqrt{\frac{\omega_1^2 I_1 + \omega_2^2 I_2}{I_1 + I_2}} //$$

Youtube / Arihant Senior

4.51. A uniform rod of length l performs small oscillations about the horizontal axis OO' perpendicular to the rod and passing through one of its points. Find the distance between the centre of inertia of the rod and the axis OO' at which the oscillation period is the shortest. What is it equal to?

Let CoM be at distance x from hanging point.



$$\omega = \sqrt{\frac{C}{I_0}} = \sqrt{\frac{Mgx}{\frac{Ml^2}{12} + Mx^2}} = \sqrt{\frac{g}{\frac{l^2}{12} + x}} \xrightarrow{\text{maximum when}} x = \sqrt{\left(\frac{l^2}{12}\right)} = \frac{l}{2\sqrt{3}} //$$

$$\tau_0 = \underbrace{Mgx \sin \theta}_C$$

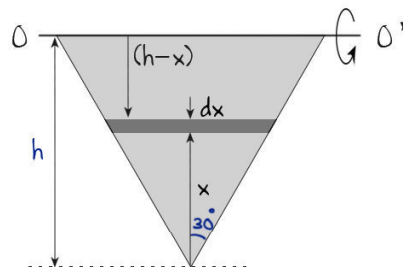
$$Ax + \frac{B}{x}$$

is minimum when $Ax = \frac{B}{x}$
i.e., $x = \sqrt{\frac{B}{A}}$

i.e., ω is maximum

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4.52. A thin uniform plate shaped as an equilateral triangle with a height h performs small oscillations about the horizontal axis coinciding with one of its sides. Find the oscillation period and the reduced length of the given pendulum.



$$dI_{OO'} = dm (h-x)^2$$

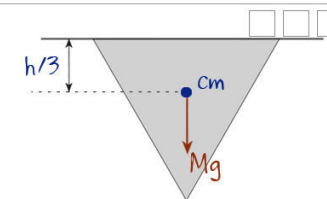
$$= \frac{2x \tan 30^\circ dx}{h^2/\sqrt{3}} M (h-x)^2$$

Area of strip
Area of eq. triangle

$$\Rightarrow I_{OO'} = \frac{2M}{h^2} \int_0^h x(h-x)^2 dx$$

$$= \frac{Mh^2}{6}$$

$$\begin{aligned} \text{A } \therefore T &= 2\pi \sqrt{\frac{I_{OO'}}{C}} \\ &= 2\pi \sqrt{\frac{Mh^2/6}{\frac{Mgh}{3}}} \\ &= 2\pi \sqrt{\frac{h}{2g}} // \end{aligned}$$



If plate is disturbed by θ

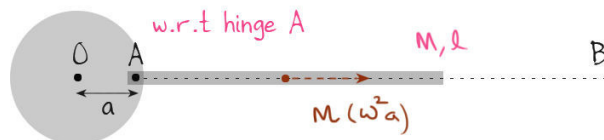
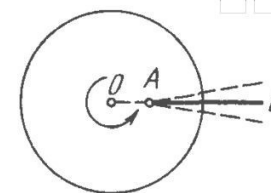
$$\tau_{OO'} = \underbrace{Mgh \frac{h}{3} \theta}_C$$

$$\text{B Comparing with: } 2\pi \sqrt{\frac{l}{g}}$$

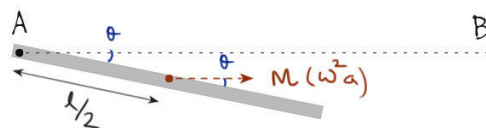
$$\text{Reduced length} = h/2 //$$

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4.53. A smooth horizontal disc rotates about the vertical axis O (Fig. 4.15) with a constant angular velocity ω . A thin uniform rod AB of length l performs small oscillations about the vertical axis A fixed to the disc at a distance a from the axis of the disc. Find the frequency ω_0 of these oscillations.



When disturbed slightly about AB:

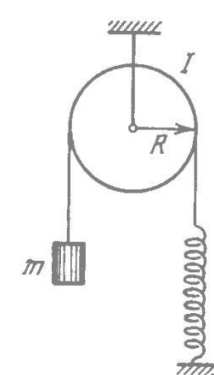


Restoring torque: $\tau_A = (M(\omega^2 a) \frac{l}{2}) \theta$

$$\begin{aligned} \therefore \omega_0 &= \sqrt{\frac{C}{I}} \\ &= \sqrt{\frac{M\omega^2 a \frac{l}{2}}{\left(\frac{Ml^2}{3}\right)}} \\ &= \omega \sqrt{\frac{3a}{2l}} \end{aligned}$$

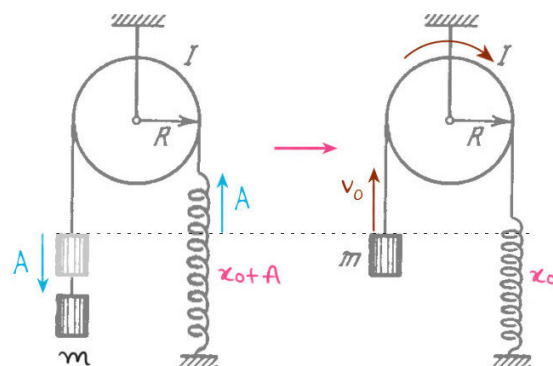
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4.54. Find the frequency of small oscillations of the arrangement illustrated in Fig. 4.16. The radius of the pulley is R , its moment of inertia relative to the rotation axis is I , the mass of the body is m , and the spring stiffness is k . The mass of the thread and the spring is negligible, the thread does not slide over the pulley, there is no friction in the axis of the pulley.



Extension in spring at mean position is given by: $mg = kx_0$ (1)

Lets pull the block further by A and release. Its velocity as it comes to mean position be v_0



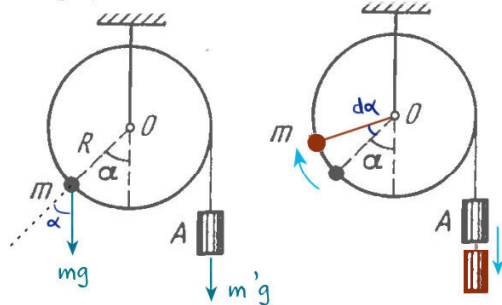
$$E_{\text{extreme}} = E_{\text{mean}}$$

$$\begin{aligned} -mgA + \frac{1}{2}k(x_0 + A)^2 &= \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2 + \frac{1}{2}I\left(\frac{v_0}{R}\right)^2 \\ \Rightarrow -mgA + \frac{1}{2}kx_0^2 + \frac{1}{2}kA^2 + kx_0A &= \frac{1}{2}kx_0^2 + \frac{v_0^2}{2}\left(m + \frac{I}{R^2}\right) \end{aligned}$$

$$\Rightarrow v_0^2 = \frac{kA^2}{m + I/R^2} \quad \text{comparing with} \quad \omega = \sqrt{\frac{k}{m + I/R^2}}$$

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4.55. A uniform cylindrical pulley of mass M and radius R can freely rotate about the horizontal axis O (Fig. 4.17). The free end of a thread tightly wound on the pulley carries a deadweight A . At a certain angle α it counterbalances a point mass m fixed at the rim of the pulley. Find the frequency of small oscillations of the arrangement.



Let deadweight A be m'

Initially, net torque about $O = 0$

$$\therefore (mg \sin \alpha) R = m' g R$$

$$\Rightarrow m' = m \sin \alpha$$

Restoring torque on turning by $d\alpha$:

$$\tau_0 = mg \sin(\alpha + d\alpha) R - m' g R$$

$$= mgR [\sin(\alpha + d\alpha) - \sin \alpha]$$

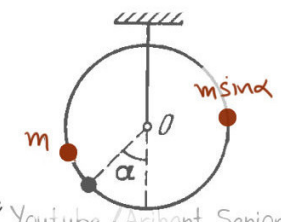
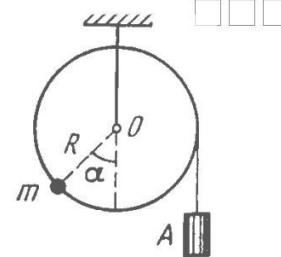
$$= mgR \left[2 \cos\left(\alpha + \frac{d\alpha}{2}\right) \sin \frac{d\alpha}{2} \right]$$

$$\approx mgR \left[2 (\cos \alpha) \frac{d\alpha}{2} \right]$$

$$= mgR \cos \alpha d\alpha$$

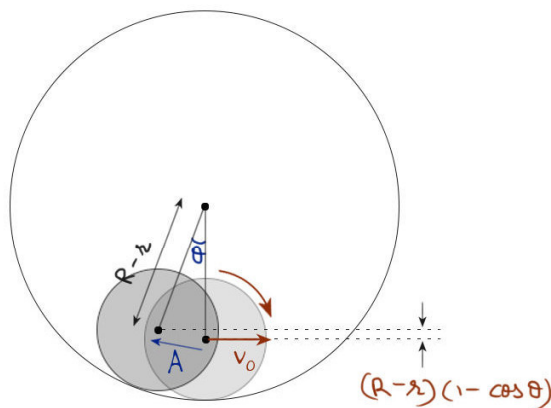
Lets stick A on the side to resemble a physical pendulum.

$$\therefore \omega = \sqrt{\frac{C}{I}} = \sqrt{\frac{mgR \cos \alpha}{\frac{MR^2}{2} + mR^2 + m \sin^2 \alpha R^2}}$$



Youtube / Arihant Senior

4.56. A solid uniform cylinder of radius r rolls without sliding along the inside surface of a cylinder of radius R , performing small oscillations. Find their period.



$$A = (R-r) \theta$$

Lets displace the cylinder by A and release. Angle turned be θ and velocity of com, when it returns to mean position be v_0 .

$$E_{\text{extreme}} = E_{\text{mean}}$$

$$mg(R-r)(1-\cos \theta) = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} m v_0^2$$

$$\Rightarrow mg(R-r) 2 \sin^2 \frac{\theta}{2} = \frac{1}{2} \left(\frac{mR^2}{2} \right) \left(\frac{v_0}{R} \right)^2 + \frac{1}{2} m v_0^2$$

$$\Rightarrow \frac{g}{2(R-r)} \left[(R-r) \theta \right]^2 = \frac{3}{4} v_0^2$$

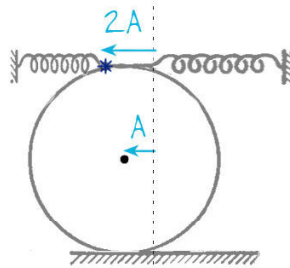
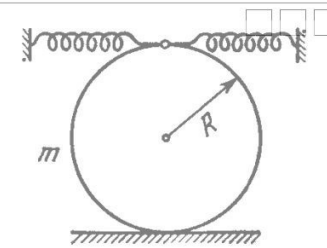
$$\Rightarrow v_0^2 = \frac{2g}{3(R-r)} A^2$$

comparing with $v_0 = \omega A$

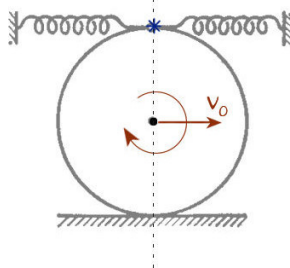
$$\omega = \sqrt{\frac{2g}{3(R-r)}}$$

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4.57. A solid uniform cylinder of mass m performs small oscillations due to the action of two springs whose combined stiffness is equal to k (Fig. 4.18). Find the period of these oscillations in the absence of sliding.



Lets displace the cylinder com by A and release.
Velocity when it comes back to mean position be v_0 .



$$E_{\text{extreme}} = E_{\text{mean}}$$

$$\frac{1}{2} k (2A)^2 = \frac{1}{2} I_{\text{cm}} \left(\frac{v_0}{R} \right)^2 + \frac{1}{2} m v_0^2$$

$$\Rightarrow 2kA^2 = \frac{3}{4} m v_0^2$$

$$\Rightarrow v_0^2 = \frac{8k}{3m} A^2$$

comparing with $v_0 = \omega A$

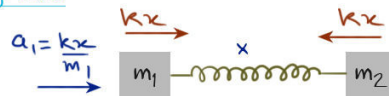
$$\omega = \sqrt{\frac{8k}{3m}} //$$

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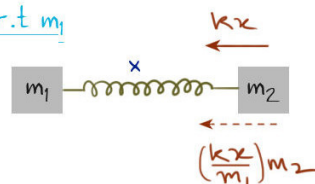
4.58. Two cubes with masses m_1 and m_2 were interconnected by a weightless spring of stiffness k and placed on a smooth horizontal surface. Then the cubes were drawn closer to each other and released simultaneously. Find the natural oscillation frequency of the system.

Let the extension in spring be x .

w.r.t. ground



w.r.t m_1



w.r.t m_1 extension in m_2 is x

Even though extension in m_2 w.r.t ground is less than x .

$$\therefore \text{Restoring force on 2} = \left(\frac{kx}{m_1} \right) m_2 + kx$$

$$\Rightarrow m_2 a = \frac{k(m_2 + m_1)}{m_1} x$$

$$\Rightarrow a = \frac{k(m_2 + m_1)}{m_1 m_2} x$$

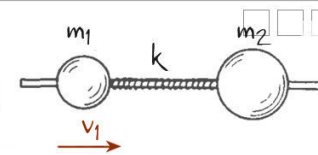
ω^2

$$\therefore \omega = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} //$$

Note: ω will be same, in all reference frames.

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4.59. Two balls with masses $m_1 = 1.0 \text{ kg}$ and $m_2 = 2.0 \text{ kg}$ are slipped on a thin smooth horizontal rod (Fig. 4.19). The balls are interconnected by a light spring of stiffness $k = 24 \text{ N/m}$. The left-hand ball is imparted the initial velocity $v_1 = 12 \text{ cm/s}$. Find:



- (a) the oscillation frequency of the system in the process of motion; Same as in 4.58:
(b) the energy and the amplitude of oscillations.

B Energy of oscillation is NOT $\frac{1}{2} m v_1^2$

It is PE of spring at maximum compression:
or KE of system w.r.t CoM at natural spring length.



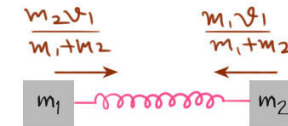
$$m_1 v_1 = (m_1 + m_2) v_{cm}$$

$$\Rightarrow v_{cm} = \frac{m_1 v_1}{m_1 + m_2}$$

$$\therefore v_{1/cm} = v_1 - v_{cm} = \frac{m_2 v_1}{m_1 + m_2}$$

$$v_{2/cm} = v_2 - v_{cm} = -\frac{m_1 v_1}{m_1 + m_2}$$

$\therefore$ w.r.t CoM @ mean position



$$E_{osc} = \frac{1}{2} m_1 \left(\frac{m_2 v_1}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1 v_1}{m_1 + m_2} \right)^2$$

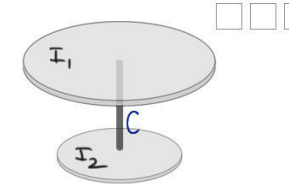
$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_1^2$$

$$= \frac{1}{2} k A^2$$

$$\Rightarrow A = \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}} v_1$$

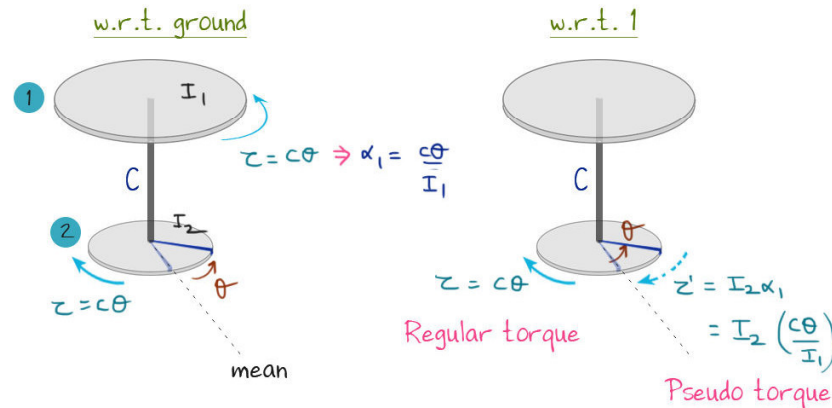
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4.60. Find the period of small torsional oscillations of a system consisting of two discs slipped on a thin rod with torsional coefficient C . The moments of inertia of the discs relative to the rod's axis are equal to I_1 and I_2 .



Pseudo Torque Similar to pseudo force problem 4.58.

Let the bottom disc is twisted by θ (keeping top disc fixed). Both discs are then released.



w.r.t 1 angle turned by 2 is θ

$\therefore$ Restoring torque on 2 w.r.t 1:

$$\tau_{2/1} = C\theta + I_2 \left(\frac{C\theta}{I_1} \right)$$

$$\Rightarrow I_2 \alpha = C \left(1 + \frac{I_2}{I_1} \right) \theta$$

$$\Rightarrow \alpha = \frac{C(I_1 + I_2)}{I_1 I_2} \theta$$

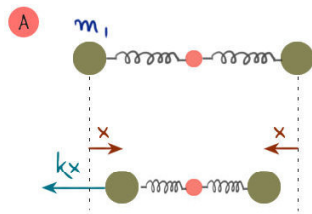
ω^2

$$\therefore \omega = \sqrt{\frac{C(I_1 + I_2)}{I_1 I_2}}$$

Note: ω will be same, in all reference frames

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4.61. A mock-up of a CO₂ molecule consists of three balls interconnected by identical light springs and placed along a straight line in the state of equilibrium. Such a system can freely perform oscillations of two types, as shown by the arrows in Fig. 4.20. Knowing the masses of the atoms, find the ratio of frequencies of these oscillations.

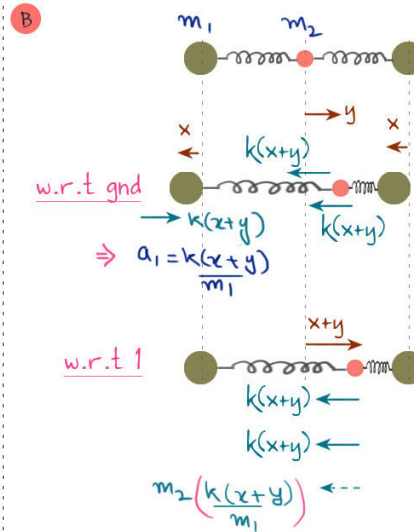


w.r.t gnd

Displacement of $m_1 = x$

Restoring force on $m_1 = kx$

$$\therefore \omega = 2\pi \sqrt{\frac{m_1}{k}}$$

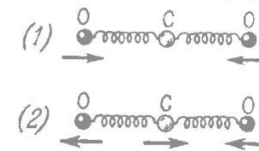


w.r.t gnd

$$\Rightarrow a_1 = \frac{k(x+y)}{m_1}$$

w.r.t 1

$$m_2 \left(\frac{k(x+y)}{m_1} \right) \leftarrow$$



$\therefore$ Restoring force on 2 wrt 1:

$$F_{2,1} = 2k(x+y) + \frac{m_2 k}{m_1}(x+y)$$

$$m_2 a = \frac{(2m_1 + m_2)k}{m_1}(x+y)$$

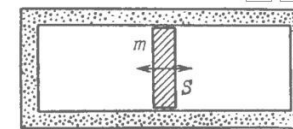
$$\Rightarrow a = \underbrace{\frac{2m_1 + m_2}{m_1 m_2}}_{\omega^2} (x+y)$$

$\therefore$ w.r.t 1 extension in 2 is $x+y$

$$\therefore \omega = \sqrt{\frac{2m_1 + m_2}{m_1 m_2}}$$

Youtube / Arihant Senior

4.62. In a cylinder filled up with ideal gas and closed from both ends there is a piston of mass m and cross-sectional area S (Fig. 4.21). In equilibrium the piston divides the cylinder into two equal parts, each with volume V_0 . The gas pressure is p_0 . The piston was slightly displaced from the equilibrium position and released. Find its oscillation frequency, assuming the processes in the gas to be adiabatic and the friction negligible.

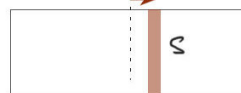
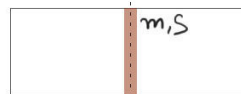


For adiabatic process:

$$pV^\gamma = \text{const}$$

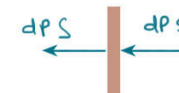
$$\Rightarrow \frac{dp}{p} + \gamma \frac{dV}{V} = 0$$

$$\Rightarrow dp = -\frac{\gamma p}{V} dV$$



$$dV = S dx \text{ both sides}$$

$$\therefore |dp| = \frac{\gamma p}{V} (S dx) \text{ both sides}$$



$\therefore$ Restoring force on piston = $2dP S$

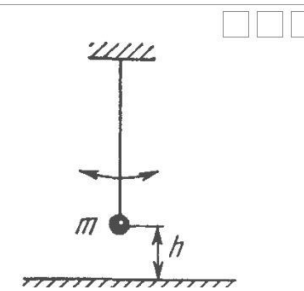
$$\Rightarrow ma = 2 \left(\frac{\gamma p S dx}{V} \right) S$$

$$\Rightarrow a = \underbrace{\frac{2\gamma p S^2}{Vm}}_{\omega^2} dx$$

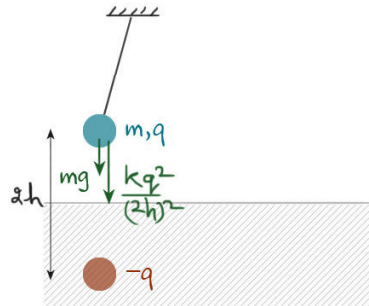
$$\therefore \omega = \sqrt{\frac{2\gamma p S^2}{Vm}}$$

Youtube / Arihant Senior

4.63. A small ball of mass $m = 21 \text{ g}$ suspended by an insulating thread at a height $h = 12 \text{ cm}$ from a large horizontal conducting plane performs small oscillations (Fig. 4.22). After a charge q had been imparted to the ball, the oscillation period changed $\eta = 2.0$ times. Find q .



When slightly disturbed, h won't change significantly



Equal charge of opposite sign is mirrored due to conducting plane

'Reduced' in this case

$$\therefore mg_{\text{eff}} = mg + \frac{kq^2}{4h^2}$$

$$\Rightarrow g_{\text{eff}} = g + \frac{kq^2}{4mh^2}$$

Given: $T_2 = \frac{1}{\eta} T_1$

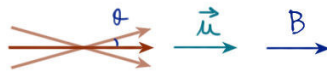
$$\Rightarrow 2\pi \sqrt{\frac{l}{g_{\text{eff}}}} = \frac{1}{\eta} 2\pi \sqrt{\frac{l}{g}}$$

$$\Rightarrow \eta^2 g = g + \frac{kq^2}{4mh^2}$$

$$\Rightarrow q = 2h \sqrt{\frac{mg(\eta^2 - 1)}{k}}$$

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4.64. A small magnetic needle performs small oscillations about an axis perpendicular to the magnetic induction vector. On changing the magnetic induction the needle's oscillation period decreased $\eta = 5.0$ times. How much and in what way was the magnetic induction changed? The oscillation damping is assumed to be negligible.



We know, $\tau = \vec{\mu} \times \vec{B}$
Initially it's at rest.

On slightly disturbing the needle by θ

$$\tau = \mu B \sin \theta$$

$$\Rightarrow I\alpha = \mu B \theta$$

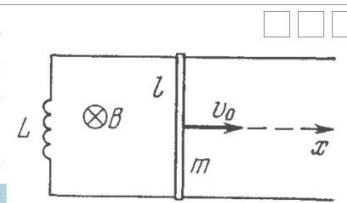
$$\Rightarrow \alpha = \underbrace{\frac{\mu B}{I}}_{\omega^2} \theta$$

Given: $\frac{2\pi}{\omega_2} = \frac{1}{\eta} \frac{2\pi}{\omega_1}$

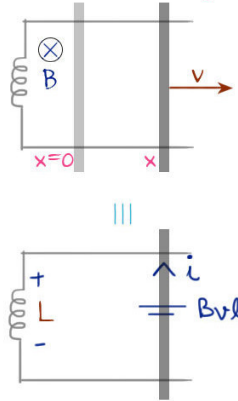
$$\Rightarrow \sqrt{\frac{I}{\mu B_2}} = \frac{1}{\eta} \sqrt{\frac{I}{\mu B_1}}$$

$$\Rightarrow B_2 = \eta^2 B_1$$

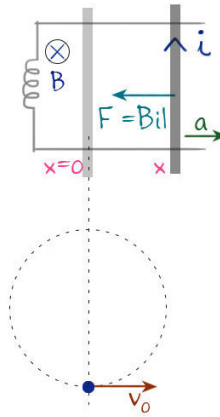
4.65. A loop (Fig. 4.23) is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field with induction B . The distance between the conductors is equal to l . At the moment $t = 0$ the rod is imparted an initial velocity v_0 directed to the right. Find the law of its motion $x(t)$ if the electric resistance of the loop is negligible.



Let the velocity be v at general x .



$$\begin{aligned} Bvl - L \frac{di}{dt} &= 0 \\ \Rightarrow B l \frac{dx}{dt} &= L \frac{di}{dt} \\ \Rightarrow B l \int_0^x dx &= L \int_0^i di \\ \Rightarrow B l x &= L i \\ \Rightarrow i &= \frac{B l}{L} x \end{aligned}$$



$\therefore$ Restoring force on rod at x :

$$m a = B i l$$

$$\Rightarrow a = \frac{B(B l x) l}{m L}$$

$$\Rightarrow a = \underbrace{\frac{B^2 l^2}{m L}}_{\omega^2} x$$

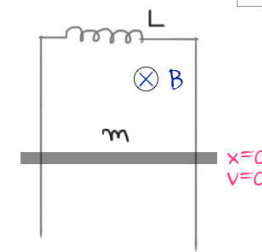
$$v_0 = \omega A$$

$$\therefore x = A \sin \omega t$$

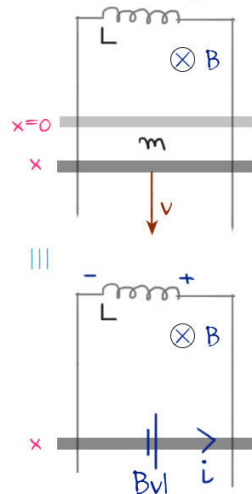
$$x = \frac{v_0}{\omega} \sin \omega t$$

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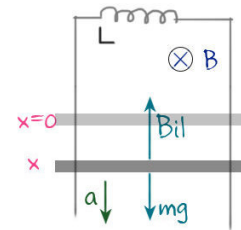
4.66. A coil of inductance L connects the upper ends of two vertical copper bars separated by a distance l . A horizontal conducting connector of mass m starts falling with zero initial velocity along the bars without losing contact with them. The whole system is located in a uniform magnetic field with induction B perpendicular to the plane of the bars. Find the law of motion $x(t)$ of the connector.



Let the velocity be v at general x .



$$\begin{aligned} Bvl - L \frac{di}{dt} &= 0 \\ \Rightarrow B l \frac{dx}{dt} &= L \frac{di}{dt} \\ \Rightarrow B l \int_0^x dx &= L \int_0^i di \\ \Rightarrow B l x &= L i \\ \Rightarrow i &= \frac{B l}{L} x \end{aligned}$$



$\therefore$ Force on rod at x :

$$m a = m g - B i l$$

$$\Rightarrow a = - \left[\frac{B(B l x) l}{m L} - g \right]$$

$$\Rightarrow a = - \underbrace{\frac{B^2 l^2}{m L}}_{\omega^2} \left(x - \frac{m L g}{B^2 l^2} \right)$$

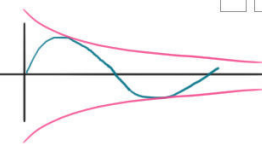
$$\therefore x = A (1 - \cos \omega t)$$

$$x_{\text{mean}} = A$$

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4.67. A point performs damped oscillations according to the law $x = a_0 e^{-\beta t} \sin \omega t$. Find:

- (a) the oscillation amplitude and the velocity of the point at the moment $t = 0$;
 (b) the moments of time at which the point reaches the extreme positions.



Amplitude

$$x = a_0 e^{-\beta t} \sin \omega t$$

$$\Rightarrow \dot{x} = a_0 e^{-\beta t} (\omega \cos \omega t - \beta \sin \omega t)$$

A $a = a_0 e^{-\beta t}$

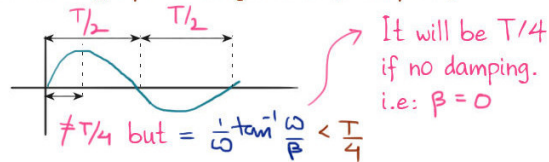
$$\Rightarrow a_{t=0} = a_0 //$$

$$\dot{x}_{t=0} = a_0 \omega //$$

Answer of B part is NOT: $\frac{T}{4} + n(\frac{T}{2})$

ie $\frac{\pi}{2\omega} + \frac{n\pi}{\omega}$

Because graph is assymetric from peak.



B At extreme positions: $\dot{x} = 0$

$$\Rightarrow \tan \omega t = \frac{\omega}{\beta}$$

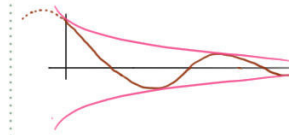
$$\Rightarrow \omega t = n\pi + \tan^{-1} \left(\frac{\omega}{\beta} \right)$$

$$\Rightarrow t = \frac{n\pi}{\omega} + \frac{1}{\omega} \tan^{-1} \left(\frac{\omega}{\beta} \right)$$

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4.68. A body performs torsional oscillations according to the law $\varphi = \varphi_0 e^{-\beta t} \cos \omega t$. Find:

- (a) the angular velocity $\dot{\varphi}$ and the angular acceleration $\ddot{\varphi}$ of the body at the moment $t = 0$;
 (b) the moments of time at which the angular velocity becomes maximum.



$$\varphi = \varphi_0 e^{-\beta t} \cos \omega t$$

$$\Rightarrow \dot{\varphi} = \varphi_0 (-\omega e^{-\beta t} \sin \omega t - \beta e^{-\beta t} \cos \omega t)$$

$$\Rightarrow \ddot{\varphi} = \varphi_0 (-\omega^2 e^{-\beta t} \cos \omega t + \omega \beta e^{-\beta t} \sin \omega t + \beta \omega e^{-\beta t} \sin \omega t + \beta^2 e^{-\beta t} \cos \omega t)$$

$$= \varphi_0 e^{-\beta t} [(\beta^2 - \omega^2) \cos \omega t + 2\beta \omega \sin \omega t]$$

A $\dot{\varphi}_{t=0} = -\beta \varphi_0 //$

$$\ddot{\varphi}_{t=0} = \varphi_0 (\beta^2 - \omega^2) //$$

B $\dot{\varphi}_{\max}$, when $\ddot{\varphi} = 0 \Rightarrow \tan \omega t = \frac{\omega^2 - \beta^2}{2\beta\omega}$

$$\Rightarrow \omega t = n\pi + \tan^{-1} \frac{\omega^2 - \beta^2}{2\beta\omega}$$

$$\Rightarrow t = \frac{n\pi}{\omega} + \frac{1}{\omega} \tan^{-1} \frac{\omega^2 - \beta^2}{2\beta\omega} //$$

$n=1,2,3$
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4.70. A point performs damped oscillations with frequency $\omega = 25 \text{ s}^{-1}$. Find the damping coefficient β if at the initial moment the velocity of the point is equal to zero and its displacement from the equilibrium position is $\eta = 1.020$ times less than the amplitude at that moment.

lets say $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

$$\Rightarrow \dot{x} = a_0 e^{-\beta t} [-\omega \sin(\omega t + \alpha) - \beta \cos(\omega t + \alpha)]$$

Given: $\dot{x}_{t=0} = 0 \Rightarrow \tan \alpha = \frac{-\beta}{\omega}$ (1)

Given: $x_{t=0} = \frac{a_0}{\eta}$

$$\Rightarrow a_0 \cos \alpha = \frac{a_0}{\eta}$$

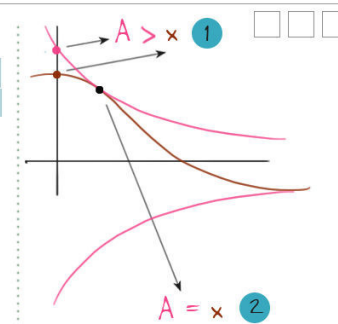
$$\Rightarrow \cos \alpha = \frac{1}{\eta} \quad (2)$$

$\because \beta > 0, \alpha$ lies in 4th quadrant

From 1: $\tan^2 \alpha = \frac{\beta^2}{\omega^2}$

$$\omega^2 (\sec^2 \alpha - 1) = \beta^2$$

$$\Rightarrow \beta = \omega \sqrt{\eta^2 - 1} //$$

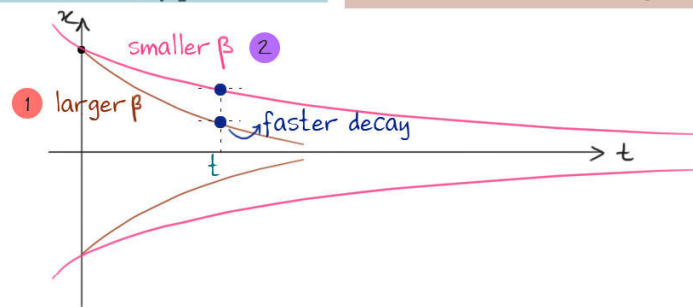


1 Amplitude A is slightly higher than displacement x at peaks.

2 They are equal little ahead of peaks.

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4.72. There are two damped oscillations with the following periods T and damping coefficients β : $T_1 = 0.10 \text{ ms}, \beta_1 = 100 \text{ s}^{-1}$ and $T_2 = 10 \text{ ms}, \beta_2 = 10 \text{ s}^{-1}$. Which of them decays faster?



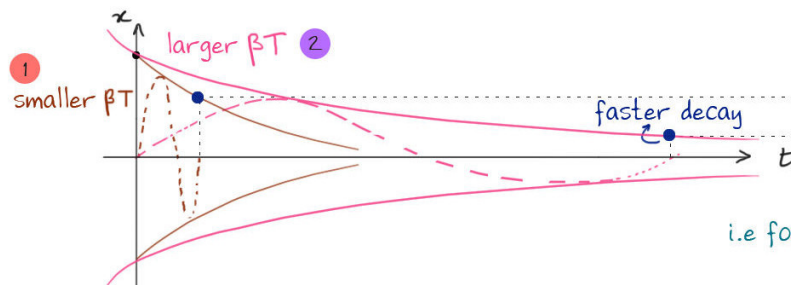
$$\text{Let } A_1 = a e^{-\beta_1 t}$$

$$A_2 = a e^{-\beta_2 t}$$

In terms of absolute time t, oscillation with larger β decays faster.

i.e for given t: $e^{-\beta_1 t} < e^{-\beta_2 t}$

$\therefore$ 1 decays faster at given t.



But in terms of Time period (or phase), oscillation with larger βT decays faster.

i.e for one time period: $e^{-\beta_2 T_2} < e^{-\beta_1 T_1}$

$\therefore$ 2 decays faster in one time period.

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4.73. A mathematical pendulum oscillates in a medium for which the logarithmic damping decrement is equal to $\lambda_0 = 1.50$. What will be the logarithmic damping decrement if the resistance of the medium increases $n = 2.00$ times? How many times has the resistance of the medium to be increased for the oscillations to become impossible?

We know, $\lambda = \beta T = \beta \frac{2\pi}{\omega} = \beta \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{(\frac{\omega_0}{\beta})^2 - 1}}$

A $\lambda_0 = \frac{2\pi}{\sqrt{(\frac{\omega_0}{\beta_0})^2 - 1}} \quad (1)$

$\lambda_f = \frac{2\pi}{\sqrt{(\frac{\omega_0}{n\beta_0})^2 - 1}} \quad (2)$

Eliminating $\frac{\omega_0}{\beta_0}$ from 1,2:

$\lambda_f = \frac{2\pi \lambda_0 n}{\sqrt{4\pi^2 + \lambda_0^2 (1-n^2)}} //$

B Lets say resistance β_0 is increased n_c times.
Oscillations become impossible at critical damping.

$\therefore$ at critical damping: $\omega_0 = n_c \beta_0$ ($n_c = ?$)

$\Rightarrow n_c = \frac{\omega_0}{\beta_0} = \sqrt{\left(\frac{2\pi}{\lambda_0}\right)^2 + 1}$

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4.74. A deadweight suspended from a weightless spring extends it by $\Delta x = 9.8$ cm. What will be the oscillation period of the deadweight when it is pushed slightly in the vertical direction? The logarithmic damping decrement is equal to $\lambda = 3.1$.

$\lambda = \beta T \quad (1)$

$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \quad (2)$

$mg = k \Delta x \Rightarrow \frac{k}{m} = \omega_0^2 = \frac{g}{\Delta x} \quad (3)$

From 1,2,3:

$T = \frac{2\pi}{\sqrt{\frac{g}{\Delta x} - \frac{\lambda^2}{T^2}}}$

$\Rightarrow T^2 \left(\frac{g}{\Delta x} - \frac{\lambda^2}{T^2} \right) = 4\pi^2$

$\Rightarrow T = \sqrt{\left(4\pi^2 + \lambda^2 \right) \frac{\Delta x}{g}} //$

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4.77. Find the quality factor of a mathematical pendulum $l = 50$ cm long if during the time interval $\tau = 5.2$ min its total mechanical energy decreases $\eta = 4.0 \cdot 10^4$ times.

We know, $E = \frac{1}{2} k A^2$

At t : $E = \frac{1}{2} k (A_0 e^{-\beta t})^2$ ①

At $t + \tau$: $\frac{E}{\eta} = \frac{1}{2} k (A_0 e^{-\beta(t+\tau)})^2$ ②

Divide 1 and 2:

$\eta = e^{2\beta\tau}$

$\Rightarrow \beta = \frac{\ln \eta}{2\tau}$

$\therefore Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T}$

$= \frac{\pi}{\beta \cdot \frac{2\pi}{\omega'}}$ $\leftarrow \omega' = \sqrt{\omega_0^2 - \beta^2}$

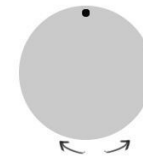
$= \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta}$

$= \frac{1}{2} \sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$ $\leftarrow \text{For pendulum } \omega_0 = \sqrt{\frac{g}{l}}$

$= \frac{1}{2} \sqrt{\frac{g^4 \tau^2}{l \ln^2 \eta} - 1}$

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4.78. A uniform disc of radius $R = 13$ cm can rotate about a horizontal axis perpendicular to its plane and passing through the edge of the disc. Find the period of small oscillations of that disc if the logarithmic damping decrement is equal to $\lambda = 1.00$.



$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$

$\lambda = \beta T$

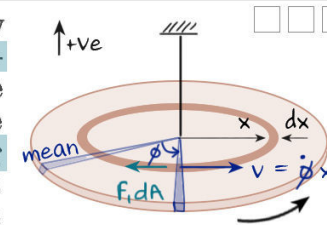
$\Rightarrow T = \frac{2\pi}{\sqrt{\frac{2g}{3R} - \lambda^2/T^2}}$

$\omega_0 = \sqrt{\frac{mg l_{cm}}{I}} = \sqrt{\frac{mgR}{\frac{3}{2}mR^2}} = \sqrt{\frac{2g}{3R}}$

$\Rightarrow T = \sqrt{\frac{3R}{2g} (4\pi^2 + \lambda^2)}$

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4.79. A thin uniform disc of mass m and radius R suspended by an elastic thread in the horizontal plane performs torsional oscillations in a liquid. The moment of elastic forces emerging in the thread is equal to $N = \alpha\phi$, where α is a constant and ϕ is the angle of rotation from the equilibrium position. The resistance force acting on a unit area of the disc is equal to $F_1 = \eta v$, where η is a constant and v is the velocity of the given element of the disc relative to the liquid. Find the frequency of small oscillation.



Lets say disc has rotated by ϕ and its angular velocity is $\dot{\phi}$

Elastic torque: $N_1 = -\alpha\phi$

Viscous torque: $dN_2 = -(F_1 dA) \times$
 $= -(\eta v \cdot 2\pi x dx) \times$
 $= -(\eta \dot{\phi} x \cdot 2\pi x dx) \times$
 $\Rightarrow N_2 = -2\pi\eta\dot{\phi} \int_0^R x^3 dx$
 $= -\frac{\pi\eta R^4}{2} \dot{\phi}$

$$N_{\text{net}} = I\ddot{\phi}$$

$$\Rightarrow I\ddot{\phi} = N_1 + N_2$$

$$\Rightarrow \frac{mR^2}{2} \ddot{\phi} = -\alpha\phi - \frac{\pi\eta R^4}{2} \dot{\phi}$$

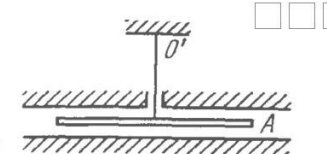
$$\Rightarrow \ddot{\phi} + \frac{\pi\eta R^2}{m} \dot{\phi} + \frac{2\alpha}{mR^2} \phi = 0$$

$$\therefore \omega_0^2 = \frac{2\alpha}{mR^2}, \beta = \frac{1}{2} \left(\frac{\pi\eta R^2}{m} \right)$$

$$\Rightarrow \omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{\frac{2\alpha}{mR^2} - \left(\frac{\pi\eta R^2}{2m} \right)^2}$$

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4.80. A disc A of radius R suspended by an elastic thread between two stationary planes (Fig. 4.24) performs torsional oscillations about its axis OO' . The moment of inertia of the disc relative to that axis is equal to I , the clearance between the disc and each of the planes is equal to h , with $h \ll R$. Find the viscosity of the gas surrounding the disc A if the oscillation period of the disc equals T and the logarithmic damping decrement, λ .



We know, viscous force per unit area is: $F_1 = \eta \frac{dv}{dz}$

Elastic torque: $N_1 = -\alpha\phi$

Viscous torque:

$dN_2 = -(F_1 dA) \times$ For top and bottom
 $= -\left(\eta \frac{\dot{\phi} x}{h} \cdot 2(2\pi x dx) \right) \times$
 $\Rightarrow N_2 = -\frac{4\pi\eta}{h} \dot{\phi} \int_0^R x^3 dx = -\frac{\pi\eta R^4}{h} \dot{\phi}$

$$N_{\text{net}} = I\ddot{\phi} = N_1 + N_2$$

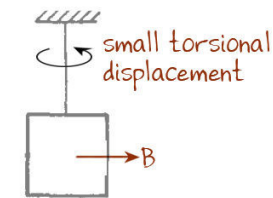
$$\Rightarrow I\ddot{\phi} + \frac{\eta\pi R^4}{h} \dot{\phi} + \alpha\phi = 0$$

$$\therefore \beta = \frac{1}{2} \left(\frac{\eta\pi R^4}{hI} \right) = \frac{\lambda}{T}$$

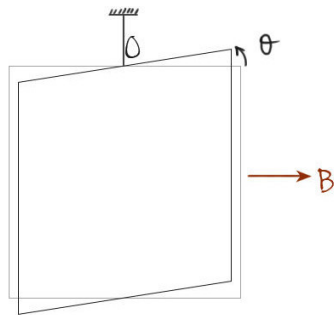
$$\Rightarrow \eta = \frac{2\lambda h I}{\pi R^4 T}$$

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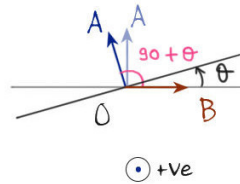
4.81. A conductor in the shape of a square frame with side a suspended by an elastic thread is located in a uniform horizontal magnetic field with induction B . In equilibrium the plane of the frame is parallel to the vector B (Fig. 4.25). Having been displaced from the equilibrium position, the frame performs small oscillations about a vertical axis passing through its centre. The moment of inertia of the frame relative to that axis is equal to I , its electric resistance is R . Neglecting the inductance of the frame, find the time interval after which the amplitude of the frame's deviation angle decreases e-fold.



Lets say frame is twisted by small angle θ , and its angular velocity is $\dot{\theta}$



Front view:



Top view:

Sol: Resistive torque calculation:

$$\text{Flux: } \phi = \vec{B} \cdot \vec{A} = BA \cos(90 + \theta) \approx -BA\theta$$

$$i = -\frac{\dot{\phi}}{R} = -\frac{BA\dot{\theta}}{R}$$

$$\otimes |N_2| = \vec{u} \times \vec{B} = i (\vec{A} \times \vec{B}) = \frac{BA\dot{\theta}}{R} AB \sin(90 + \theta) \approx \frac{A^2 B^2 \dot{\theta}}{R}$$

$$\text{Restoring torque: } \otimes |N_1| = C\theta$$

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$$\begin{aligned} \text{For the frame: } I\alpha &= N_{\text{net}} \\ &= -|N_1| - |N_2| \\ &= -C\theta - \frac{B^2 A^2 \dot{\theta}}{R} \end{aligned}$$

$$\Rightarrow \alpha + \frac{B^2 A^2}{IR} \dot{\theta} + \frac{C}{I} \theta = 0$$

(standard damped SHM equation)

$$\therefore \beta = \frac{1}{2} \left(\frac{B^2 A^2}{IR} \right)$$

$$\text{Given: If, } A = A_0 e^{-\beta t}$$

$$\frac{A}{e} = A_0 e^{-\beta(t+z)}$$

$$\text{dividing} \rightarrow e = e^{\beta z}$$

$$\Rightarrow z = \frac{1}{\beta} = \frac{2IR}{B^2 A^2} = \frac{2IR}{B^2 a^4}$$

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A

$t_1 = \pi \sqrt{\frac{m}{k}}$

$t_2 = \pi \sqrt{\frac{m}{k}}$

- We know,

- 
- A diagram of a mass-spring system is shown on the left. It consists of a brown rectangular mass at the bottom, connected to a coiled spring, which is attached to a horizontal line representing a ceiling. To the right of the diagram, the formula for the period of oscillation is written in brown ink: $T = 2\pi \sqrt{\frac{m}{k}}$.

-
- Diagram illustrating a mass-spring system. A mass is suspended from a ceiling by a spring. A dashed line indicates the 'new mean' position. A downward arrow labeled mg represents the gravitational force.

$$T = t_1 + t_2 = 2\pi \sqrt{\frac{m}{k}}$$

And block will start only if:

$$kx \geq f \Rightarrow x \geq t$$

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- Sol: From diagram, its clear that, between any two consecutive stops, blocks distance from 0 decreases 2t further.

$$t = \frac{\mu mg}{k}$$



mean position when
block moves left

Natural length of spring

∴ Stationary block at: beginning: After 1st half cycle: After 2nd half cycle: After nth half cycle:	distance from 0 x_0 $x_0 - 2t$ $x_0 - 4t$ ⋮ $x_0 - 2nt$
---	---

- For half oscillations to continue:

$$x_0 - 2nt \geq t$$

$$\Rightarrow n \leq \frac{1}{2} \left(\frac{x_0}{t} - 1 \right)$$

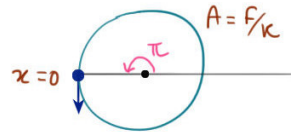
$$\Rightarrow n \leq \frac{1}{2}(15-1) \Rightarrow n \leq 7 \Rightarrow 7 \text{ half cycles}$$

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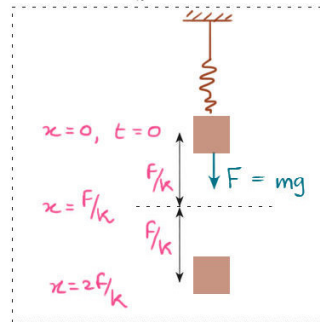
4.84. A particle of mass m can perform undamped harmonic oscillations due to an electric force with coefficient k . When the particle was in equilibrium, a permanent force F was applied to it for τ seconds. Find the oscillation amplitude that the particle acquired after the action of the force ceased. Draw the approximate plot $x(t)$ of oscillations. Investigate possible cases.

$t < \tau$

Its similar to a block dropped, when spring is at natural length. It will perform SHM about new mean at F/k .



∴ equation of motion: $x - x_{\text{mean}} = A \cos(\omega t + \pi)$
 i.e. $x = \frac{F}{k} (1 - \cos \omega t)$ (1)



Like spring

$$x - x_{\text{mean}} = A' \cos[\omega(t - \tau) + \phi] \quad (2)$$

$$\Rightarrow \dot{x} = -A' \omega \sin[\omega(t - \tau) + \phi]$$

From 1 & 2, at $t = \tau$

$$x = \frac{F}{k} (1 - \cos \omega \tau) = A' \cos \phi$$

$$\& \dot{x} = \frac{F}{k} \omega \sin \omega \tau = -A' \omega \sin \phi$$

Square and add: $\left(\frac{F}{k}\right)^2 (2 - 2 \cos \omega \tau) = A'^2$

$$\Rightarrow A' = \frac{2F}{k} \left| \sin \frac{\omega \tau}{2} \right|$$

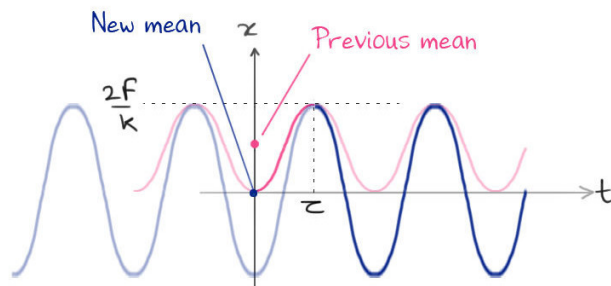
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$$A' = \frac{2F}{k} \sin \frac{\omega \tau}{2} \quad (3)$$

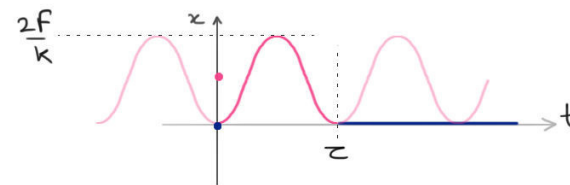
- > New amplitude A' depends on τ
- > Time period is same as before though: $\frac{2\pi}{\omega}$
- > x & $\dot{x}$ (slope) will be same at τ for both the graphs.

Different cases based on different values of τ

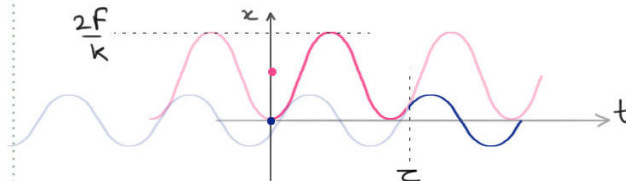
A A' is maximum ($2F/k$) when $|\sin \frac{\omega \tau}{2}| = 1$



B A' is minimum (0) when $|\sin \frac{\omega \tau}{2}| = 0$



C Any other time, A' will be between 0 and $2F/k$.



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4.89. Due to the external vertical force $F_x = F_0 \cos \omega t$ a body suspended by a spring performs forced steady-state oscillations according to the law $x = a \cos(\omega t - \phi)$. Find the work performed by the force F during one oscillation period.

Method 1

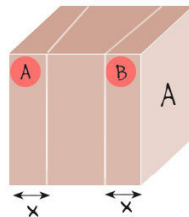
$$\begin{aligned}
 W &= \int F \cdot dx \\
 x &= a \cos(\omega t - \phi) \\
 dx &= -a\omega \sin(\omega t - \phi) dt \\
 &= \int F_0 \cos \omega t (-a\omega) \sin(\omega t - \phi) dt \\
 &= -F_0 a \omega \int (\cos \omega t \sin \omega t \cos \phi - \cos^2 \omega t \sin \phi) dt \\
 &= -\frac{F_0 a \omega}{2} \int_0^{2\pi/\omega} (\cancel{\cos \omega t \sin \omega t} \cos \phi - \cancel{\cos^2 \omega t} \sin \phi - \sin \phi) dt \\
 &= -\frac{F_0 a \omega}{2} (-\sin \phi) \left(\frac{2\pi}{\omega} \right) = \pi F_0 a \sin \phi //
 \end{aligned}$$

Method 2

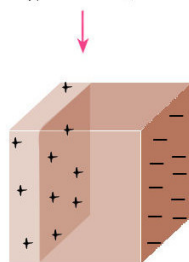
$$\begin{aligned}
 \text{Power } P &= F \cdot v \\
 \text{Work} &= \int_0^T P dt \\
 &= \int F v dt
 \end{aligned}$$

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4.94. Due to a certain cause the free electrons in a plane copper plate shifted over a small distance x at right angles to its surface. As a result, a surface charge and a corresponding restoring force emerged, giving rise to so-called plasma oscillations. Find the angular frequency of these oscillations if the free electron concentration in copper is $n = 0.85 \cdot 10^{29} \text{ m}^{-3}$



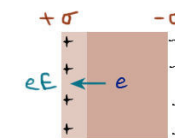
e are evenly distributed



e 's shifted right by x .

$\therefore$ A region will have no electrons.
And B region electrons will be collected on right surface.

Sol:



Restoring

Force: $= eE$

$$\begin{aligned}
 &= e \frac{\sigma}{\epsilon_0} \rightarrow \sigma = \frac{nV \cdot e}{A} = \frac{n(Ax)e}{A} \\
 &= nex
 \end{aligned}$$

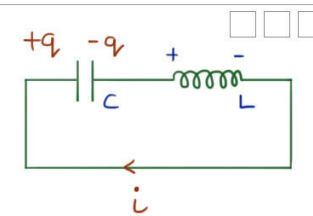
$$= e \left(\frac{nex}{\epsilon_0} \right)$$

$$= \frac{ne^2}{\epsilon_0} x$$

$$\therefore \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{ne^2}{\epsilon_0 m_e}} //$$

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4.95. An oscillating circuit consisting of a capacitor with capacitance C and a coil of inductance L maintains free undamped oscillations with voltage amplitude across the capacitor equal to V_m . For an arbitrary moment of time find the relation between the current I in the circuit and the voltage V across the capacitor. Solve this problem using Ohm's law and then the energy conservation law.



Method 1 By Ohm's law

$$i = \frac{dq}{dt}$$

$$\text{KVL: } -\frac{q}{C} - L \frac{di}{dt} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \underbrace{\left(\frac{1}{LC}\right)}_{\omega^2} q = 0$$

$$\therefore q = q_0 \cos(\omega t + \phi)$$

$$\xrightarrow{CV} CV_m$$

$$\Rightarrow V = V_m \cos(\omega t + \phi)$$

$$\text{diff} \rightarrow i = -CV_m \omega \sin(\omega t + \phi)$$

sq and add:

$$\left(\frac{i}{C\omega}\right)^2 + V^2 = V_m^2 \Rightarrow Li^2 + CV^2 = CV_m^2 //$$

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Method 2

$$\text{By energy conservation: } \frac{1}{2} CV^2 + \frac{1}{2} Li^2 = \text{constant} \quad (1)$$

$$\text{When } V \text{ across capacitor is maximum: } \frac{dV}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{q}{C}\right) = 0 \Rightarrow \frac{1}{C} \frac{dq}{dt} = 0 \Rightarrow i = 0$$

$$\therefore \text{From 1: } \frac{1}{2} CV_m^2 + \frac{1}{2} Li^2 = \text{constant} \quad (2)$$

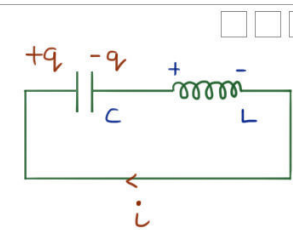
$$\text{From 1 and 2: } \frac{1}{2} CV^2 + \frac{1}{2} Li^2 = \frac{1}{2} CV_m^2 //$$

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4.96. An oscillating circuit consists of a capacitor with capacitance C , a coil of inductance L with negligible resistance, and a switch. With the switch disconnected, the capacitor was charged to a voltage V_m and then at the moment $t = 0$ the switch was closed.

Find:

- the current $I(t)$ in the circuit as a function of time;
- the emf of self-inductance in the coil at the moments when the electric energy of the capacitor is equal to that of the current in the coil.



A From 4.95: $q = q_0 \cos(\omega t + \phi)$

$$\begin{aligned} & \xrightarrow{CV} CV_m \\ \Rightarrow V &= V_m \cos(\omega t + \phi) \xrightarrow[t=0]{V=V_m} \phi = 0 \\ \text{diff} \rightarrow \dot{q} &= -CV_m \omega \sin(\omega t + \phi) \xrightarrow{\therefore} i = -CV_m \omega \sin(\omega t) \quad \text{with } \omega = \frac{1}{\sqrt{LC}} \end{aligned}$$

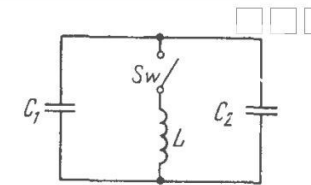
B $\frac{1}{2} CV_m^2 = (\frac{1}{2} L i^2) \times 2 \Rightarrow i = \frac{V_m}{\sqrt{L}}$

emf across coil $\left| L \frac{di}{dt} \right| = \text{emf across capacitor } V = \frac{V_m}{\sqrt{2}}$

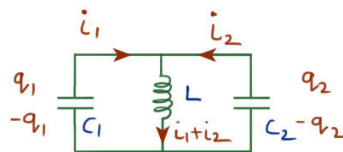
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4.98. In an oscillating circuit shown in Fig. 4.27 the coil inductance is equal to $L = 2.5 \text{ mH}$ and the capacitors have capacitances $C_1 = 2.0 \mu\text{F}$ and $C_2 = 3.0 \mu\text{F}$. The capacitors were charged to a voltage $V = 180 \text{ V}$, and then the switch Sw was closed. Find:

- the natural oscillation frequency;
- the peak value of the current flowing through the coil.



Method 1



Step 1

$$\frac{q_1}{C_1} = L \frac{d(i_1 + i_2)}{dt} = \frac{q_2}{C_2} \quad \text{on differentiating:} \quad \frac{i_1}{C_1} = \frac{i_2}{C_2}$$

$$\frac{q_1}{C_1} - L \frac{d(i_1 + i_2)}{dt} = 0$$

$$\Rightarrow \frac{q_1}{C_1} - L \frac{d}{dt} \left(i_1 + \frac{C_2}{C_1} i_1 \right) \Rightarrow q_1 + L(C_1 + C_2) \frac{d^2 q_1}{dt^2} = 0 \Rightarrow \frac{d^2 q_1}{dt^2} + \frac{1}{L(C_1 + C_2)} q_1 = 0$$

Also $\dot{q}_1 = -\frac{dq_1}{dt}$

Step 1 Differential eqn from KVL

Step 2 Get eqn in single variable

Step 3 Solve eqn and put conditions

A $\therefore \omega = \frac{1}{\sqrt{L(C_1 + C_2)}}$

Step 2

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Step 3 $q_1 = q_{10} \cos(\omega t + \phi)$ (As capacitor is charged to maximum initially)

$$\Rightarrow \dot{q}_1 = -q_{10} \omega \sin(\omega t)$$

Similarly: $\dot{q}_2 = -q_{20} \omega \sin(\omega t)$

$$\begin{aligned} \Rightarrow i &= \dot{q}_1 + \dot{q}_2 = -(q_{10} + q_{20}) \omega \sin \omega t \\ &= -(C_1 V + C_2 V) \frac{1}{\sqrt{L(C_1 + C_2)}} \sin \omega t \end{aligned}$$

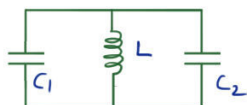
$$= -V \sqrt{\frac{C_1 + C_2}{L}} \sin \omega t$$

B Maximum i

(Note: Both q_1 and q_2 will be zero at maximum i , i.e. maximum i_1 and i_2)

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Method 2 If we open L , capacitors act in parallel with $C_{eq} = C_1 + C_2$.



A $\omega = \frac{1}{\sqrt{L C_{eq}}} = \frac{1}{\sqrt{L(C_1 + C_2)}}$

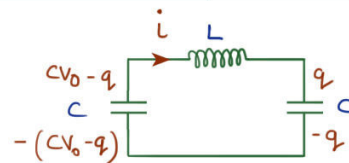
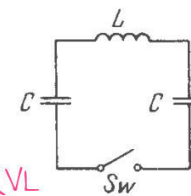
B $i_{max} = q_0 \omega$

$$= C_{eq} V \omega$$

$$= (C_1 + C_2) V \frac{1}{\sqrt{L(C_1 + C_2)}} = V \sqrt{\frac{C_1 + C_2}{L}} //$$

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4.99. An electric circuit shown in Fig. 4.28 has a negligibly small active resistance. The left-hand capacitor was charged to a voltage V_0 and then at the moment $t = 0$ the switch Sw was closed. Find the time dependence of the voltages in left and right capacitors.



Step 1

$$\frac{CV_0 - q}{C} - L \frac{di}{dt} - \frac{q}{C} = 0 \quad i = \frac{dq}{dt}$$

$$\Rightarrow V_0 - \frac{2q}{C} - L \frac{d^2q}{dt^2} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \underbrace{\frac{2}{LC}}_{\omega^2} \left(q - \underbrace{\frac{CV_0}{2}}_{\text{Mean value of } q} \right) = 0 \quad \text{Step 2}$$

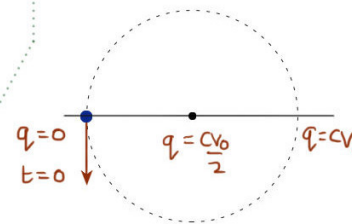
Step 1 Differential eqn from KVL

Step 2 Get eqn in single variable

Step 3 Solve eqn and put conditions

Step 3

$$\therefore q = \frac{CV_0}{2} (1 - \cos \omega t)$$



$$\text{For right capacitor: } V(t) = q/C = \frac{V_0}{2} (1 - \cos \omega t)$$

$$\text{For left capacitor: } V(t) = \frac{CV_0 - q}{C} = V_0 - \frac{q}{C} = \frac{V_0}{2} (1 + \cos \omega t)$$

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4.100. An oscillating circuit consists of an inductance coil L and a capacitor with capacitance C . The resistance of the coil and the lead wires is negligible. The coil is placed in a permanent magnetic field so that the total flux passing through all the turns of the coil is equal to Φ . At the moment $t = 0$ the magnetic field was switched off. Assuming the switching off time to be negligible compared to the natural oscillation period of the circuit, find the circuit current as a function of time t .

Flux CAN NOT change suddenly.

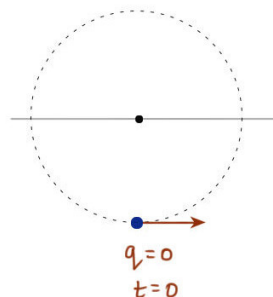
$\therefore$ immediate current i_0 will flow to maintain Φ

$$\therefore \Phi = Li_0$$

At general time t :

$$-\frac{q}{C} - L \frac{di}{dt} = 0 \quad i = \frac{dq}{dt}$$

$$\Rightarrow \frac{d^2q}{dt^2} + \left(\frac{1}{LC} \right) q = 0 \quad (1)$$

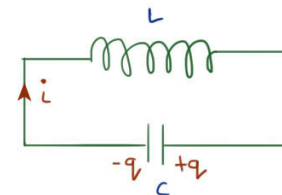
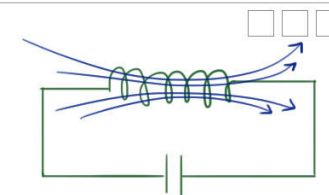


Solution of 1:

$$q = q_0 \sin \omega t$$

$$\Rightarrow i = \underbrace{q_0 \omega}_{i_0 = \Phi/L} \cos \omega t$$

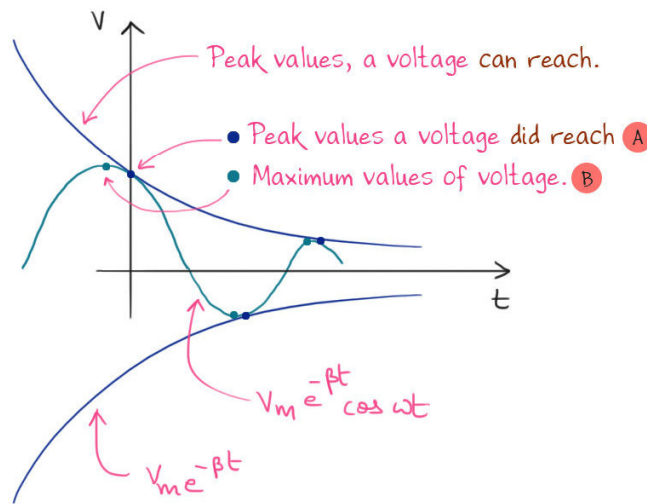
$$\Rightarrow i = \frac{\Phi}{L} \cos \left(\frac{1}{\sqrt{LC}} t \right)$$



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4.101. The free damped oscillations are maintained in a circuit, such that the voltage across the capacitor varies as $V = V_m e^{-\beta t} \cos \omega t$. Find the moments of time when the modulus of the voltage across the capacitor reaches

- peak values;
- maximum (extremum) values.



Sol: $V = V_m e^{-\beta t} \cos \omega t$

A At points, $|V| = V_m e^{-\beta t}$

$$\Rightarrow V_m e^{-\beta t} |\cos \omega t| = V_m e^{-\beta t}$$

$$\Rightarrow \cos \omega t = \pm 1$$

$$\Rightarrow \omega t = n\pi //$$

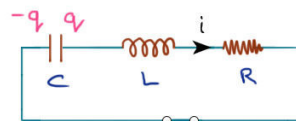
B At points, $\frac{dV}{dt} = 0$

$$\Rightarrow V_m e^{-\beta t} [-\beta \cos \omega t + (-\omega) \sin \omega t] = 0$$

$$\Rightarrow \tan \omega t = -\frac{\beta}{\omega} \Rightarrow \omega t = n\pi + \tan^{-1}\left(-\frac{\beta}{\omega}\right)$$

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4.102. A certain oscillating circuit consists of a capacitor with capacitance C , a coil with inductance L and active resistance R , and a switch. When the switch was disconnected, the capacitor was charged; then the switch was closed and oscillations set in. Find the ratio of the voltage across the capacitor to its peak value at the moment immediately after closing the switch.



$$\frac{q}{C} - L \frac{di}{dt} - iR = 0$$

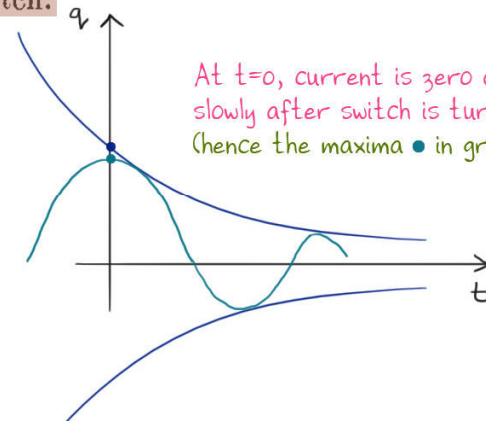
$$i \rightarrow -\frac{dq}{dt}$$

$$\Rightarrow \frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \left(\frac{1}{LC}\right) q = 0$$

solution

$$q = q_m e^{-\beta t} \cos(\omega t + \phi), \quad \beta = \frac{R}{2L}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$\omega_0 = \frac{1}{\sqrt{LC}}$



At $t=0$, current is zero as it will start slowly after switch is turned on. (hence the maxima • in graph at $t=0$)

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Sol: $q = q_m e^{-\beta t} \cos(\omega t + \phi)$

$$\Rightarrow \frac{dq}{dt} = q_m e^{-\beta t} [-\beta \cos(\omega t + \phi) - \omega \sin(\omega t + \phi)]$$

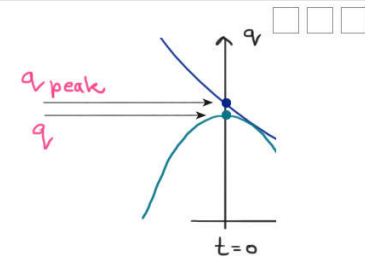
At $t=0$, current is zero

$$\therefore \left. \frac{dq}{dt} \right|_{t=0} = -\beta \cos \phi - \omega \sin \phi = 0 \Rightarrow \tan \phi = -\frac{\beta}{\omega}$$

$$\text{Required ratio} = \left. \frac{V}{V_{\text{peak}}} \right|_{t=0} = \frac{q/q}{q_{\text{peak}}/q} \Big|_{t=0} = \frac{q_m e^{-\beta t} \cos(\omega t + \phi)}{q_m e^{-\beta t}} = \cos \phi = \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

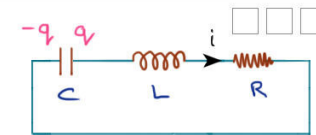
$\downarrow \tilde{\omega} = \omega_0^2 - \beta^2$

$$\sqrt{1 - \frac{\beta^2}{\omega_0^2}} \quad \beta = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$



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4.103. A circuit with capacitance C and inductance L generates free damped oscillations with current varying with time as $I = I_m e^{-\beta t} \sin \omega t$. Find the voltage across the capacitor as a function of time, and in particular, at the moment $t = 0$.



$$\frac{q}{C} - L \frac{di}{dt} - iR = 0$$

$$\therefore V_C = \frac{q}{C} = L \frac{di}{dt} + iR$$

$i = i_m e^{-\beta t} \sin \omega t$

$$= L i_m e^{-\beta t} (-\beta \sin \omega t + \omega \cos \omega t + 2\beta \sin \omega t)$$

$$= L i_m e^{-\beta t} \sqrt{\omega^2 + \beta^2} \left(\frac{\beta \sin \omega t + \omega \cos \omega t}{\sqrt{\omega^2 + \beta^2}} \right)$$

$\omega^2 = \omega_0^2 - \beta^2$

$$= L i_m e^{-\beta t} \omega_0 \sin(\omega t + \alpha), \quad \alpha = \tan^{-1} \frac{\omega}{\beta}$$

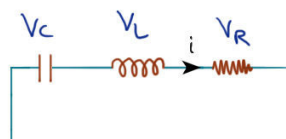
$\omega_0 = \frac{1}{\sqrt{LC}}$

$$= i_m \sqrt{\frac{L}{C}} e^{-\beta t} \sin(\omega t + \alpha) \xrightarrow{t=0} V_C \Big|_{t=0} = i_m \sqrt{\frac{L}{C}} \sin \alpha = i_m \sqrt{\frac{L}{C}} \frac{\omega}{\omega_0}$$

$\rightarrow \frac{1}{\sqrt{LC}}$

$= i_m L \omega$
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4.104. An oscillating circuit consists of a capacitor with capacitance $C = 4.0 \mu\text{F}$ and a coil with inductance $L = 2.0 \text{ mH}$ and active resistance $R = 10 \Omega$. Find the ratio of the energy of the coil's magnetic field to that of the capacitor's electric field at the moment when the current has the maximum value.



For voltage drops V_C , V_L , and V_R :

$$\text{KVL: } V_C + V_L + V_R = 0$$

Sol: As current is maximum: $\frac{di}{dt} = 0$

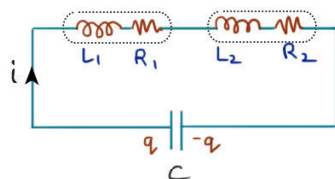
$$\Rightarrow V_L = L \frac{di}{dt} = 0$$

$$\Rightarrow |V_C| = V_R = iR$$

$$\therefore \frac{E_L}{E_C} = \frac{\frac{1}{2} Li^2}{\frac{1}{2} C V_C^2} = \frac{Li^2}{C (iR)^2} = \frac{L}{CR^2} //$$

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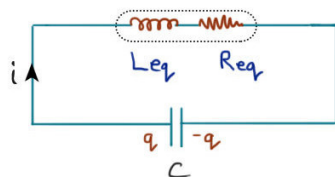
4.105. An oscillating circuit consists of two coils connected in series whose inductances are L_1 and L_2 , active resistances are R_1 and R_2 , and mutual inductance is negligible. These coils are to be replaced by one, keeping the frequency and the quality factor of the circuit constant. Find the inductance and the active resistance of such a coil.



|||

$$\text{KVL: } \frac{q}{C} - L_1 \frac{di}{dt} - iR_1 - L_2 \frac{di}{dt} - iR_2 = 0$$

$$\Rightarrow \frac{q}{C} - (L_1 + L_2) \frac{di}{dt} - (R_1 + R_2) i = 0 \quad (1)$$



$$\text{KVL: } \frac{q}{C} - L_{eq} \frac{di}{dt} - R_{eq} i = 0 \quad (2)$$

Comparing 1 and 2 if, $L_{eq} = L_1 + L_2$ and $R_{eq} = R_1 + R_2$

Then solution of equations (and therefore ω and Q value) will be same.

Note: $\therefore$ Equivalent L and R are in series, so can be just added.

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4.106. How soon does the current amplitude in an oscillating circuit with quality factor $Q = 5000$ decrease $\eta = 2.0$ times if the oscillation frequency is $\nu = 2.2$ MHz?

In a series LCR circuit: $Q = \frac{\omega L}{R}$ $\left\{ Q = \frac{\pi}{\beta T} = \frac{\pi}{\frac{R}{2L} \frac{2\pi}{\omega}} = \frac{\omega L}{R} \right\}$

Sol: $i = i_m e^{-\beta t}$

$\frac{i}{\eta} = i_m e^{-\beta(t+\tau)}$

divide $\rightarrow \eta = e^{\beta \tau}$

$\Rightarrow \tau = \frac{\ln \eta}{\beta} = \frac{Q \ln \eta}{\pi \nu}$ //

$\beta = \frac{R}{2L} = \frac{\omega}{2Q} = \frac{2\pi \nu}{2Q} = \frac{\pi \nu}{Q}$

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4.107. An oscillating circuit consists of capacitance $C = 10 \mu\text{F}$, inductance $L = 25 \text{ mH}$, and active resistance $R = 1.0 \Omega$. How many oscillation periods does it take for the current amplitude to decrease e-fold?

$i = i_m e^{-\beta t}$

$\frac{i}{e} = i_m e^{-\beta(t+nT)}$

dividing $\rightarrow e = e^{\beta nT}$

$\Rightarrow n = \frac{1}{\beta T}$

$= \frac{1}{\beta \frac{2\pi}{\omega_0}} = \frac{\omega_0}{2\pi \beta}$

$= \frac{1}{2\pi} \sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$

$= \frac{1}{2\pi} \sqrt{\frac{4L}{CR^2} - 1}$ //

$\omega_0 \rightarrow \frac{1}{\sqrt{LC}}, \beta \rightarrow \frac{R}{2L}$

$\left(\frac{\omega_0}{\beta}\right)^2 = \frac{4L}{CR^2}$

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4.108. How much (in per cent) does the free oscillation frequency ω of a circuit with quality factor $Q = 5.0$ differ from the natural oscillation frequency ω_0 of that circuit?

$$\text{fractional difference} = \left| \frac{\omega_0 - \omega}{\omega_0} \right|$$

$$= 1 - \frac{\omega}{\omega_0} \rightarrow \sqrt{\omega^2 + \beta^2}$$

$$= 1 - \frac{1}{\sqrt{1 + \left(\frac{\beta}{\omega}\right)^2}}$$

$$= 1 - \left(1 + \frac{1}{4Q^2}\right)^{-1/2}$$

$$\approx \frac{1}{8Q^2}$$

$$Q = \frac{\omega L}{R} = \frac{\omega}{2\beta}$$

$$\Rightarrow \frac{\beta}{\omega} = \frac{1}{2Q}$$

$$\frac{1}{4Q^2} = \frac{1}{4 \cdot 5^2} = \frac{1}{100} \ll 1$$

$$\therefore \% \text{ difference} = \frac{100}{8Q^2}$$

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4.109. In a circuit shown in Fig. 4.29 the battery emf is equal to $V = 2.0$ V, its internal resistance is $r = 9.0 \Omega$, the capacitance of the capacitor is $C = 10 \mu\text{F}$, the coil inductance is $L = 100$ mH, and the resistance is $R = 1.0 \Omega$. At a certain moment the switch Sw was disconnected.

Find the energy of oscillations in the circuit

- immediately after the switch was disconnected;
- $t = 0.30$ s after the switch was disconnected.

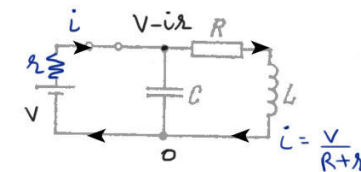
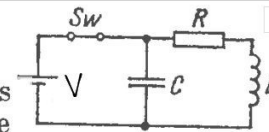
- A At $t=0$, after disconnecting the switch, current i through inductor and charge accumulated on capacitor will not change suddenly.

Energy of oscillation will be due to both

$$\therefore E_0 = \frac{1}{2} C (V - iR)^2 + \frac{1}{2} Li^2 \quad \left(i \rightarrow \frac{V}{R+r} \right)$$

$$= \frac{1}{2} C \left(V - \frac{VR}{R+r} \right)^2 + \frac{1}{2} L \left(\frac{V}{R+r} \right)^2$$

$$= \frac{1}{2} C \frac{V^2 R^2}{(R+r)^2} + \frac{1}{2} L \frac{V^2}{(R+r)^2} = \frac{V^2 (CR^2 + L)}{2(R+r)^2}$$



- B Total oscillation energy of LCR circuit is of the form of:

$$E = \frac{1}{2} k A^2$$

$$= \frac{1}{2} k (A_0 e^{-\beta t})^2$$

$$= \frac{1}{2} k A_0^2 e^{-\frac{Rt}{L}} = E_0 e^{-\frac{Rt}{L}}$$

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4.110. Damped oscillations are induced in a circuit whose quality factor is $Q = 50$ and natural oscillation frequency is $\nu_0 = 5.5$ kHz. How soon will the energy stored in the circuit decrease $\eta = 2.0$ times?

$$Q = \frac{\pi}{\beta T} = \frac{\pi}{\beta \left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{2\beta} = \frac{1}{2} \sqrt{\frac{\omega_0^2 - \beta^2}{\beta^2}} = \frac{1}{2} \sqrt{\left(\frac{\omega_0^2}{\beta^2}\right) - 1}$$

$$\Rightarrow \beta = \frac{\omega_0}{\sqrt{4Q^2 + 1}}$$

From 4.109:

$$E = E_0 e^{-2\beta t}$$

Given:

$$\frac{E}{\eta} = E_0 e^{-2\beta(t+z)}$$

dividing $\eta = e^{2\beta z}$

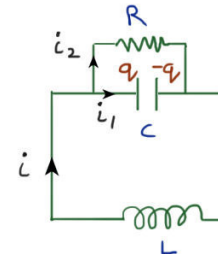
$$\Rightarrow z = \frac{\ln \eta}{2\beta} = \frac{\ln \eta \sqrt{4Q^2 + 1}}{2(2\pi\nu_0)} \approx \frac{Q \ln \eta}{2\pi\nu_0}$$

Youtube / Arihant Senior

4.111. An oscillating circuit incorporates a leaking capacitor. Its capacitance is equal to C and active resistance to R . The coil inductance is L . The resistance of the coil and the wires is negligible. Find:

- (a) the damped oscillation frequency of such a circuit;
(b) its quality factor.

Internal resistance of capacitor (R) is in parallel with capacitor



A $i = i_1 + i_2$
 $= \frac{dq}{dt} + \frac{q}{RC}$

$$L \frac{di}{dt} = -\frac{q}{C}$$

$$\Rightarrow L \left(\frac{d^2 q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} \right) = -\frac{q}{C}$$

$$\Rightarrow \frac{d^2 q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{LC} q = 0$$

$$\therefore \omega = \sqrt{\omega_0^2 - \beta^2}$$

$$= \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$$

B $Q \neq \frac{\omega L}{R}$ (as its not a series LCR circuit)

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\pi}{\beta} \frac{\omega}{2\pi} = \frac{\omega}{2\beta}$$

$$= \frac{1}{2} \sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1} = \frac{1}{2} \sqrt{\frac{4R^2 C}{L} - 1}$$

$$\omega_0 \rightarrow \frac{1}{\sqrt{LC}}, \beta \rightarrow \frac{1}{2RC}$$

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4.112. Find the quality factor of a circuit with capacitance $C = 2.0 \mu\text{F}$ and inductance $L = 5.0 \text{ mH}$ if the maintenance of undamped oscillations in the circuit with the voltage amplitude across the capacitor being equal to $V_m = 1.0 \text{ V}$ requires a power $\langle P \rangle = 0.10 \text{ mW}$. The damping of oscillations is sufficiently low.

□ □ □

We know, for damped oscillation: $\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = 0$

β is small

Solution is: $x = x_0 e^{-\beta t} \sin(\omega' t + \phi)$, where, $\omega' = \sqrt{\omega_0^2 - \beta^2}$
Amplitude (A)

∴ Total energy of damped oscillation is given by: $E = \frac{1}{2} k A^2 = \frac{1}{2} k x_0^2 e^{-2\beta t} = E_0 e^{-2\beta t}$

Sol: $E = E_0 e^{-2\beta t} \approx E_0 (1 - 2\beta t)$ ($\because e^{-x} \approx 1 - x$ for small x)

$$\Rightarrow | \langle P \rangle | = \frac{dE}{dt} = -2E_0 \beta = -2 \left(\frac{1}{2} C V_m^2 \right) \left(\frac{R}{2L} \right) \xrightarrow{R = \frac{1}{Q} \sqrt{\frac{L}{C}}} = \frac{C V_m^2}{2L} \frac{1}{Q} \sqrt{\frac{L}{C}} = \frac{V_m^2}{2Q} \sqrt{\frac{C}{L}}$$

$$\Rightarrow Q = \frac{V_m^2}{2 \langle P \rangle \sqrt{\frac{C}{L}}} //$$

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4.113. What mean power should be fed to an oscillating circuit with active resistance $R = 0.45 \Omega$ to maintain undamped harmonic oscillations with current amplitude $I_m = 30 \text{ mA}$?

□ □ □

Method 1 Power to be fed = power loss = P

$$E = E_0 e^{-2\beta t}$$

$$\therefore P = \left. \frac{dE}{dt} \right|_{t \rightarrow 0} = E_0 \frac{d}{dt} e^{-2\beta t} \approx E_0 \frac{d(1 - 2\beta t)}{dt}$$

$$\Rightarrow |P| = 2E_0 \beta = 2 \left(\frac{1}{2} L I_m^2 \right) \frac{R}{2L} = \frac{I_m^2 R}{2} //$$

Method 2

$$\text{Power loss} = i_{\text{rms}}^2 R$$

$$= \left(\frac{I_m}{\sqrt{2}} \right)^2 R$$

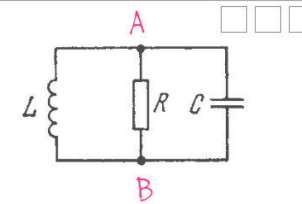
$$= \frac{I_m^2 R}{2} //$$

4.114. An oscillating circuit consists of a capacitor with capacitance $C = 1.2 \text{ nF}$ and a coil with inductance $L = 6.0 \mu\text{H}$ and active resistance $R = 0.50 \Omega$. What mean power should be fed to the circuit to maintain undamped harmonic oscillations with voltage amplitude across the capacitor being equal to $V_m = 10 \text{ V}$?

$$\begin{aligned} |P| &= 2E_0 \beta \\ &= 2 \left(\frac{1}{2} C V_m^2 \right) \frac{R}{2L} \\ &= \frac{C V_m^2 R}{2L} // \end{aligned}$$

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4.115. Find the damped oscillation frequency of the circuit shown in Fig. 4.30. The capacitance C , inductance L , and active resistance R are supposed to be known. Find how must C , L , and R be interrelated to make oscillations possible.



Just like 4.111 when L, R, C are all in parallel. $\omega = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$

For oscillations to be possible: $\omega > 0$

$$\Rightarrow \frac{1}{LC} > \frac{1}{4R^2C^2}$$

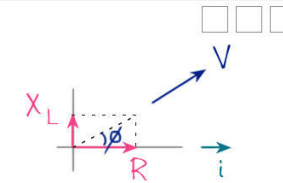
$$\Rightarrow R > \frac{1}{2} \sqrt{\frac{L}{C}}$$

If R is infinite, AB is open. And its perfect LC oscillations.

If R is zero, AB is shorted, so no oscillations at all.

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4.120. A long one-layer solenoid tightly wound of wire with resistivity ρ has n turns per unit length. The thickness of the wire insulation is negligible. The cross-sectional radius of the solenoid is equal to a . Find the phase difference between current and alternating voltage fed to the solenoid with frequency ν .



$$\tan \phi = \frac{X_L}{R} = \frac{\omega L}{R} \quad (1)$$

In RL circuit: i lags V by ϕ

Sol: Let diameter of wire be d , and length of solenoid be l .

$$\tan \phi = \frac{\omega L}{R} \quad (1)$$

$\omega L = \mu_0 n^2 V \rightarrow \pi a^2 l$
 $\frac{3 \cancel{2} \pi a n l}{A} \rightarrow \pi \frac{d^2}{4} \rightarrow \frac{1}{n} \quad (2)$

$$= \frac{\cancel{2} \pi \nu \mu_0 n^2 \pi a^2 l}{\cancel{3} \cancel{2} \pi a n l} \cdot \frac{\pi}{4 n^2} = \frac{\pi^2 \mu_0 \nu a}{4 n^2}$$

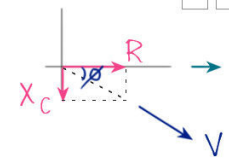
$$n = \frac{\text{no. of turns}}{\text{length}} = \frac{(l/d)}{l} = \frac{1}{d} \quad (2)$$

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4.121. A circuit consisting of a capacitor and an active resistance $R = 110 \Omega$ connected in series is fed an alternating voltage with amplitude $V_m = 110 \text{ V}$. In this case the amplitude of steady-state current is equal to $I_m = 0.50 \text{ A}$. Find the phase difference between the current and the voltage fed.

$$\tan \phi = \frac{1}{\omega C R} \quad (1) \quad I_m = \frac{V_m}{Z} = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

$$\Rightarrow \frac{1}{\omega C} = \sqrt{\frac{V_m^2}{I_m^2} - R^2} \quad (2)$$



$$\tan \phi = \frac{X_C}{R} = \frac{1}{\omega C R}$$

In RC circuit: i leads V by ϕ

From 1,2:

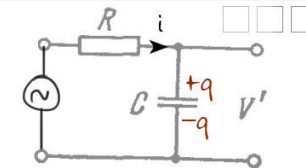
$$\phi = \tan^{-1} \sqrt{\left(\frac{V_m}{I_m R}\right)^2 - 1}$$

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4.122. Fig. 4.32 illustrates the simplest ripple filter. A voltage $V = V_0 (1 + \cos \omega t)$ is fed to the left input. Find:

(a) the output voltage $V'(t)$;

(b) the magnitude of the product RC at which the output amplitude of alternating voltage component is $\eta = 7.0$ times less than the direct voltage component, if $\omega = 314 \text{ s}^{-1}$.



Sol: A

Our AC source in a RC circuit is: $V_0 (1 + \cos \omega t)$

$$\therefore V_0 (1 + \cos \omega t) - iR - \frac{q}{C} = 0$$

$$\Rightarrow \frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R} (1 + \cos \omega t)$$

$$\Rightarrow \frac{d}{dt}(q - CV_0) + \frac{1}{RC}(q - CV_0) = \frac{V_0}{R} \cos \omega t \quad (1)$$

$\therefore (1) \approx (1)$ Solution of 1 is hence:

$$\Rightarrow (2) \approx (2) \quad \frac{q - CV_0}{C} = V_0 \left(\frac{X_C}{Z}\right) \cos(\omega t - \phi) \quad (2)$$

$$\Rightarrow V' = \frac{q}{C} = V_0 + \frac{V_0}{\sqrt{1 + (\omega CR)^2}} \cos(\omega t - \phi)$$

Lets say AC source in a RC circuit is: $V_0 \cos \omega t$

$$\therefore V_0 \cos \omega t - iR - \frac{q}{C} = 0$$

$$\Rightarrow \frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R} \cos \omega t \quad (1)$$

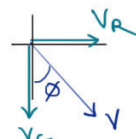
We know solution of 1 is:

$$\frac{q}{C} = V_0 \left(\frac{X_C}{Z}\right) \cos(\omega t - \phi) \quad (2)$$

$$X_C = \frac{1}{\omega C}$$

$$Z = \sqrt{\left(\frac{1}{\omega C}\right)^2 + R^2}$$

$$\phi = \tan^{-1} \frac{R}{1/\omega C}$$



$$B \quad V' = V_0 + \frac{V_0}{\sqrt{1+(\omega CR)^2}} \cos(\omega t - \phi)$$

Given: Amplitude of AC component of $V' = \left(\frac{1}{n}\right)$ Amplitude of DC component of V'

$$\therefore \frac{V_0}{\sqrt{1+(\omega CR)^2}} = \frac{1}{n} V_0$$

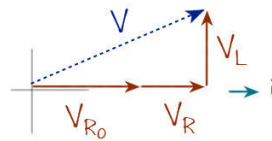
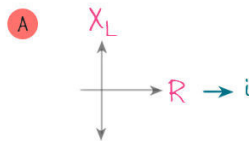
$$\Rightarrow (\omega CR)^2 = n^2 - 1$$

$$RC = \frac{\sqrt{n^2 - 1}}{\omega}$$

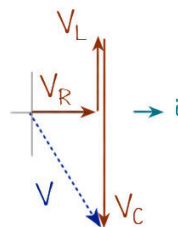
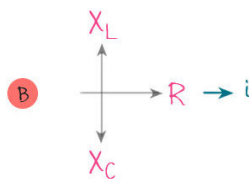
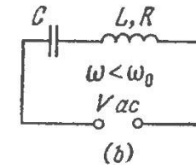
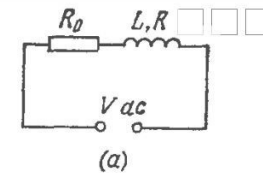
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4.123. Draw the approximate voltage vector diagrams in the electric circuits shown in Fig. 4.33 a, b. The external voltage V is assumed to be alternating harmonically with frequency ω .

In series circuit, it's convenient to keep current as reference (i.e. X-axis) as it's same across all components.



$$\begin{cases} V_L = iX_L \\ V_R = iR \\ V_{R_0} = iR_0 \end{cases}$$



$$\omega < \omega_0$$

$$\Rightarrow \omega^2 < \frac{1}{LC}$$

$$\Rightarrow \omega L < \frac{1}{\omega C}$$

$$\Rightarrow X_L < X_C$$

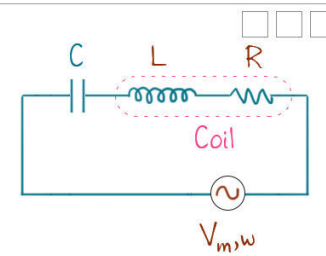
$$\Rightarrow V_L < V_C$$

$$\therefore \begin{cases} V_L = iX_L \\ V_C = iX_C \end{cases}$$

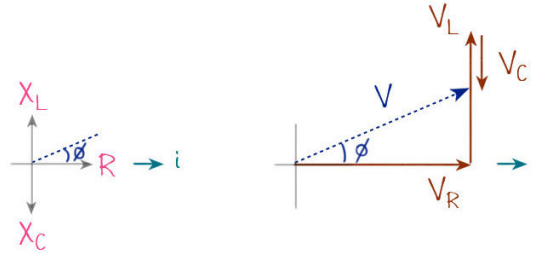
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4.124. A series circuit consisting of a capacitor with capacitance $C = 22 \mu\text{F}$ and a coil with active resistance $R = 20 \Omega$ and inductance $L = 0.35 \text{ H}$ is connected to a source of alternating voltage with amplitude $V_m = 180 \text{ V}$ and frequency $\omega = 314 \text{ s}^{-1}$. Find:

- the current amplitude in the circuit;
- the phase difference between the current and the external voltage;
- the amplitudes of voltage across the capacitor and the coil.
- If $V = V_m \cos(\omega t)$, find Voltage across capacitor and coil at time t .



B

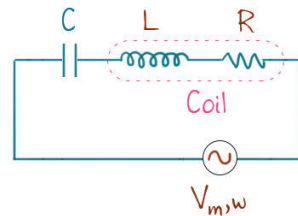


A
$$i_m = \frac{V_m}{Z} = \frac{V_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$\therefore V \text{ leads } i \text{ by } \phi = \tan^{-1} \left(\frac{X_L - X_C}{R} \right) = \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) = -60^\circ \Rightarrow i \text{ leads } V \text{ by } 60^\circ //$

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(c) the amplitudes of voltage across the capacitor and the coil.

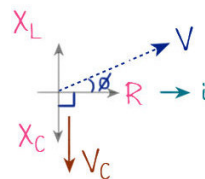


$$V_{Cm} = i_m X_C = \frac{V_m X_C}{Z} = \frac{V_m \left(\frac{1}{\omega C} \right)}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$V_{\text{coil}m} = i_m X_{\text{coil}} = \frac{V_m X_{\text{coil}}}{Z} = \frac{V_m \sqrt{R^2 + (\omega L)^2}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

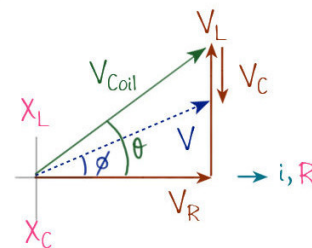
(d) If $V = V_m \cos(\omega t)$, find Voltage across capacitor and coil at time t .

Across C:



$$V_C = V_{C\max} \cos(\omega t - \phi - \frac{\pi}{2}) //$$

Across Coil:



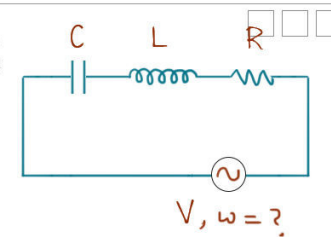
$$V_{\text{coil}} = V_{\text{coil}m} \cos(\omega t + \theta - \phi) //$$

where, $\theta = \tan^{-1} \frac{X_L}{R} = \tan^{-1} \frac{\omega L}{R}$

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4.125. A series circuit consisting of a capacitor with capacitance C , a resistance R , and a coil with inductance L and negligible active resistance is connected to an oscillator whose frequency can be varied without changing the voltage amplitude. Find the frequency at which the voltage amplitude is maximum

- (a) across the capacitor;
(b) across the coil.



A
$$V_{Cm} = \frac{V_m X_C}{Z} = V_m \frac{1/\omega C}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{V_m/C}{\sqrt{\omega^2 R^2 + (\omega^2 L - 1/C)^2}} \rightarrow \text{needs to be minimum}$$

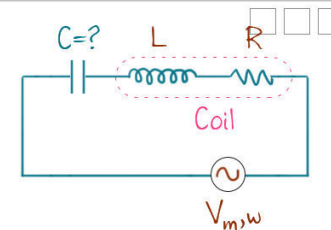
$$\therefore \frac{dV_{Cm}}{d\omega} = 0 \Rightarrow 2\omega R^2 + (\omega^2 L - 1/C)(2\omega L) = 0 \Rightarrow \omega = \sqrt{\frac{1}{L} \left(\frac{1}{C} - \frac{R^2}{2L} \right)}$$

B
$$V_{Lm} = \frac{V_m}{Z} X_L = V_m \frac{\omega L}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{V_m L}{\sqrt{\frac{R^2}{\omega^2} + (L - \frac{1}{\omega^2 C})^2}} \rightarrow \text{needs to be minimum}$$

$$\frac{dV_{Lm}}{d\omega} = 0 \Rightarrow \omega = \sqrt{\frac{2}{2LC - R^2 C^2}}$$

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4.126. An alternating voltage with frequency $\omega = 314 \text{ s}^{-1}$ and amplitude $V_m = 180 \text{ V}$ is fed to a series circuit consisting of a capacitor and a coil with active resistance $R = 40 \Omega$ and inductance $L = 0.36 \text{ H}$. At what value of the capacitor's capacitance will the voltage amplitude across the coil be maximum? What is this amplitude equal to? What is the corresponding voltage amplitude across the condenser?



V_{coil} is maximum at what value of C ?

Sol:
$$V_{\text{coil}m} = V_m \frac{X_{\text{coil}}}{Z} = \frac{V_m \sqrt{R^2 + X_L^2}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

- A $\therefore$ we can vary only C ,
 V_{coil} is maximum when: $X_L = X_C$
$$\Rightarrow \omega L = \frac{1}{\omega C}$$

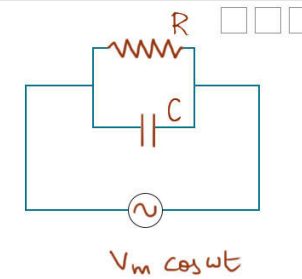
$$\Rightarrow C = \frac{1}{\omega^2 L}$$

- B Maximum value of V_{coil} is hence:
$$\frac{V_m \sqrt{R^2 + X_L^2}}{R}$$

- C Corresponding $V_C = V_m \frac{X_C}{Z} = V_m \frac{X_L}{R} = V_m \frac{\omega L}{R}$

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4.127. A capacitor with capacitance C whose interelectrode space is filled up with poorly conducting medium with active resistance R is connected to a source of alternating voltage $V = V_m \cos \omega t$. Find the time dependence of the steady-state current flowing in lead wires. The resistance of the wires is to be neglected.

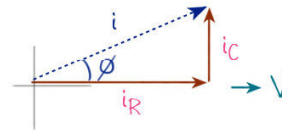


In parallel circuit, it's convenient to keep Voltage as reference (i.e. X-axis) as it's same across all components.

$$i_C = \frac{V_m}{X_C}$$

$$i_R = \frac{V_m}{R} \rightarrow V$$

$$i_L = \frac{V_m}{X_L}$$



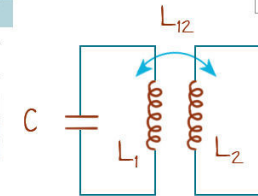
$$\phi = \tan^{-1}\left(\frac{i_C}{i_R}\right) = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}(\omega CR)$$

$$i = \sqrt{i_R^2 + i_C^2} \cos(\omega t + \phi)$$

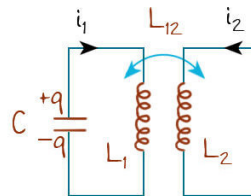
$$= V_m \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \cos(\omega t + \phi)$$

Youtube / Arihant Senior

4.128. An oscillating circuit consists of a capacitor of capacitance C and a solenoid with inductance L_1 . The solenoid is inductively connected with a short-circuited coil having an inductance L_2 and a negligible active resistance. Their mutual inductance coefficient is equal to L_{12} . Find the natural frequency of the given oscillating circuit.



KVL in both loops:



$$\phi_1 = L_1 i_1 + L_{12} i_2$$

$$\frac{q}{C} - \frac{d\phi_1}{dt} = 0$$

$$\phi_2 = L_2 i_2 + L_{12} i_1$$

$$\frac{d\phi_2}{dt} = 0$$

$$\Rightarrow \frac{q}{C} - L_1 \frac{di_1}{dt} - L_{12} \frac{di_2}{dt} = 0$$

$$\Rightarrow L_2 \frac{di_2}{dt} + L_{12} \frac{di_1}{dt} = 0$$

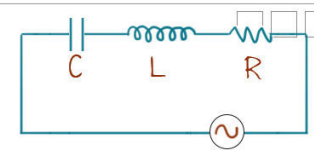
eliminating i_2 & $i_1 = -\frac{dq}{dt}$

$$\left(L_1 - \frac{L_{12}^2}{L_2}\right) \frac{d^2 q}{dt^2} + \frac{q}{C} = 0 \Rightarrow \omega = \frac{1}{\sqrt{C(L_1 - \frac{L_{12}^2}{L_2})}}$$

(Equation of SHM)

Youtube / Arihant Senior

4.129. Find the quality factor of an oscillating circuit connected in series to a source of alternating emf if at resonance the voltage across the capacitor is n times that of the source.



We know:

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\pi}{\beta \left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{2\beta} = \frac{\omega}{2\beta} = \sqrt{\frac{\omega_0^2 - \beta^2}{2\beta^2}} = \frac{1}{2} \sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1} = \frac{1}{2} \sqrt{\frac{4\omega_0^2 L^2}{R^2} - 1} \quad (1)$$

$\beta = \frac{R}{2L}$

Sol:

Given: $V_C = n V_m$

$\Rightarrow V_m \frac{X_C}{Z} = n V_m$ At resonance
 $X_C = \frac{1}{\omega_0 C}$
 $Z = R$

$\Rightarrow \frac{1/\omega_0 C}{R} = n \Rightarrow \omega_0 C R = \frac{1}{n}$ From (2) $Q = \frac{1}{2} \sqrt{4n^2 - 1}$

$L \rightarrow \frac{1}{C\omega_0^2}$

$$= \frac{1}{2} \sqrt{\frac{4}{\omega_0^2 C^2 R^2} - 1} \quad (2)$$

$\omega_0 \rightarrow \frac{1}{\sqrt{LC}}$

$$= \frac{1}{2} \sqrt{\frac{4L}{CR^2} - 1} \quad (3)$$

Youtube/Arihant Senior

4.130. An oscillating circuit consisting of a coil and a capacitor connected in series is fed an alternating emf, with coil inductance being chosen to provide the maximum current in the circuit. Find the quality factor of the system, provided an n -fold increase of inductance results in an n -fold decrease of the current in the circuit.

$$Q = \frac{1}{2} \sqrt{\frac{4\omega_0^2 L^2}{R^2} - 1} \quad (1)$$

Proved in 4.129

Maximum current occurs at resonance.

$\therefore X_C = X_L$

$\Rightarrow \frac{1}{\omega_0 C} = \omega_0 L$

Given: $\frac{1}{n} \frac{V_m}{R} = \frac{V_m}{Z}$

$\Rightarrow Z = nR$

$\Rightarrow \sqrt{R^2 + \left(\omega_0 n L - \frac{1}{\omega_0 C}\right)^2} = nR$

$\Rightarrow R^2 + \omega_0^2 L^2 (n-1)^2 = n^2 R^2$

$\Rightarrow \frac{\omega_0^2 L^2}{R^2} = \frac{n^2 - 1}{(n-1)^2} \quad (2)$

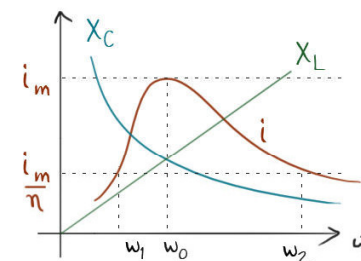
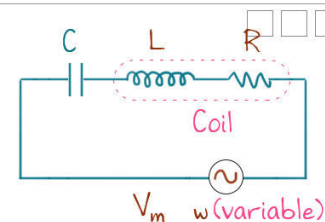
From 1,2:

$$Q = \frac{1}{2} \sqrt{\frac{4(n^2 - 1)}{(n-1)^2} - 1}$$

Youtube/Arihant Senior

4.131. A series circuit consisting of a capacitor and a coil with active resistance is connected to a source of harmonic voltage whose frequency can be varied, keeping the voltage amplitude constant. At frequencies ω_1 and ω_2 the current amplitudes are n times less than the resonance amplitude. Find:

- (a) the resonance frequency;
(b) the quality factor of the circuit.



$$\text{A} \quad \frac{1}{n} \left(\frac{V_m}{R} \right) = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

$$\Rightarrow (\frac{1}{n^2} - 1) R^2 = \left(\omega L - \frac{1}{\omega C} \right)^2 \rightarrow \text{Roots } \omega_1 \text{ and } \omega_2$$

$$\text{If } X_C > X_L \quad \sqrt{\frac{1}{n^2} - 1} R = \frac{1}{\omega_1 C} - \omega_1 L \quad (1)$$

$$\text{If } X_L > X_C \quad \sqrt{\frac{1}{n^2} - 1} R = \omega_2 L - \frac{1}{\omega_2 C} \quad (2)$$

$$(2) - (1) \Rightarrow (\omega_2 + \omega_1) L - \frac{(\omega_2 + \omega_1)}{\omega_1 \omega_2 C} = 0 \Rightarrow \omega_1 \omega_2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2} \quad (3)$$

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B From 1:

$$\sqrt{\frac{1}{n^2} - 1} R = \frac{1}{C} \left(\frac{1}{\omega_1} - \omega_1 LC \right) = \frac{1}{\omega_0^2} = \frac{1}{\omega_1 \omega_2}$$

$$\Rightarrow RC = \frac{1}{\sqrt{\frac{1}{n^2} - 1}} \left(\frac{1}{\omega_1} - \frac{1}{\omega_2} \right)$$

$$\therefore Q = \frac{1}{2} \sqrt{\frac{4}{(\omega_1 \omega_2) \frac{1}{(n^2 - 1)} \left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2} \right)^2} - 1}$$

$$= \frac{1}{2} \sqrt{\frac{4 \omega_1 \omega_2 (n^2 - 1)}{(\omega_2 - \omega_1)^2} - 1}$$

$$Q = \frac{1}{2} \sqrt{\frac{4}{\omega_0^2 C^2 R^2} - 1}$$

Proved in 4.129

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4.132. Demonstrate that at low damping the quality factor Q of a circuit maintaining forced oscillations is approximately equal to $\omega_0/\Delta\omega$, where ω_0 is the natural oscillation frequency, $\Delta\omega$ is the width of the resonance curve $I(\omega)$ at the "height" which is $1/\sqrt{2}$ times less than the resonance current amplitude. $\Rightarrow \omega' \approx \omega_0$

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\pi}{\beta \left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{2\beta} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} \xrightarrow{\text{low } \beta} \approx \frac{\omega_0}{2\beta} \quad (1)$$

Sol: Given: $\frac{1}{\sqrt{2}} \left(\frac{V_m}{R} \right) = \frac{V_m}{Z} = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}$

$$\Rightarrow 2R^2 = R^2 + (X_L - X_C)^2$$

$$\Rightarrow R^2 = (\omega L - \frac{1}{\omega C})^2$$

$$\Rightarrow \frac{R}{L} = \pm \left(\omega' - \frac{\omega_0^2}{\omega'} \right)$$

$$\Rightarrow 2\beta = \pm \left(\frac{\omega'^2 - \omega_0^2}{\omega'} \right)$$

continued...

In this problem:

3 types of frequencies

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

natural oscillation freq.

$$\omega = \sqrt{\omega_0^2 - \beta^2}$$

damped oscillation freq.

$$\omega' \text{ --- } V$$

External source freq.

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$$\Rightarrow 2\beta = \pm \frac{(\omega' + \omega_0)(\omega' - \omega_0)}{\omega'}$$

$$\Rightarrow 2\beta = \pm (\omega' - \omega_0)$$

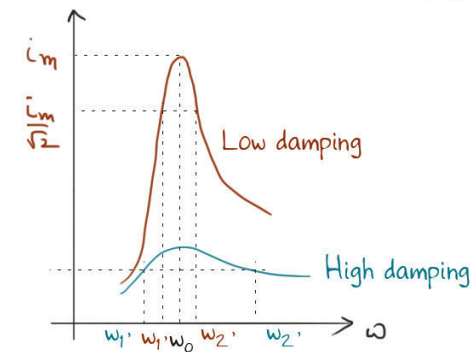
$$\Rightarrow \omega' - \omega_0 = \pm \beta$$

$$\begin{matrix} \omega_0 - \omega'_1 = \beta & \omega'_2 - \omega_0 = \beta \end{matrix}$$

adding

$$\omega'_2 - \omega'_1 = \Delta\omega = 2\beta \quad (2)$$

From 1,2: $Q = \frac{\omega_0}{\Delta\omega}$



We have already proved in 4.131 that regardless of value of β , $\omega_0 = \sqrt{\omega'_1 \omega'_2}$

And we can see in diagram too, that with low damping, as the peak becomes sharper:

$$\omega_0 = \sqrt{\omega'_1 \omega'_2} \approx \frac{\omega'_1 + \omega'_2}{2}$$

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4.133. A circuit consisting of a capacitor and a coil connected in series is fed two alternating voltages of equal amplitudes but different frequencies. The frequency of one voltage is equal to the natural oscillation frequency (ω_0) of the circuit, the frequency of the other voltage is η times higher. Find the ratio of the current amplitudes (I_0/I) generated by the two voltages if the quality factor of the system is equal to Q . Calculate this ratio for $Q = 10$ and 100 , if $\eta = 1.10$.

We know,

$$Q = \frac{1}{2} \sqrt{\frac{4\omega_0^2 L^2}{R^2} - 1}$$

(Proved in 4.129)

$$\Rightarrow \frac{\omega_0^2 L^2}{R^2} = \frac{4Q^2 + 1}{4}$$

$$\begin{aligned} \text{A} \quad \frac{I_0}{I} &= \frac{V_m/R}{V_m/Z} = \frac{Z}{R} = \frac{\sqrt{R^2 + (X_L - X_C)^2}}{R} \\ &= \sqrt{1 + \frac{1}{R^2} \left(\omega L - \frac{1}{\omega C} \right)^2} \quad \omega \rightarrow \eta \omega_0 \\ &= \sqrt{1 + \frac{\omega_0^2 L^2}{R^2} \left(\eta - \frac{1}{\eta \omega_0^2 LC} \right)^2} \\ &= \sqrt{1 + \left(\frac{4Q^2 + 1}{4} \right) \left(\eta - \frac{1}{\eta} \right)^2} // \end{aligned}$$

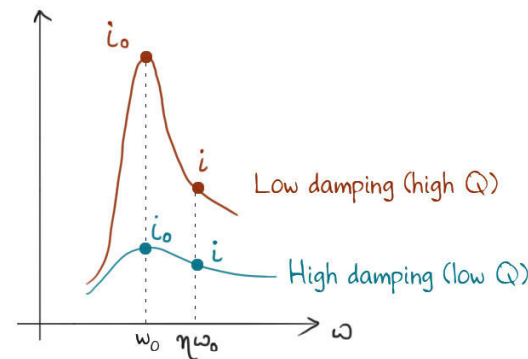
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$$\text{B} \quad \frac{I_0}{I} = \sqrt{1 + \left(\frac{4Q^2 + 1}{4} \right) \left(\eta - \frac{1}{\eta} \right)^2}$$

At $\eta = 1.1$

$$Q = 10 \rightarrow \frac{I_0}{I} = 2.2 //$$

$$Q = 100 \rightarrow \frac{I_0}{I} = 19 //$$



We can see from graph too, that at higher Q and same η , I_0/I is higher.

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4.140. A source of sinusoidal emf with constant voltage is connected in series with an oscillating circuit with quality factor $Q = 100$. At a certain frequency of the external voltage the heat power generated in the circuit reaches the maximum value. How much (in per cent) should this frequency be shifted to decrease the power generated $n = 2.0$ times?

Proved in 4.129

$$Q = \frac{1}{2} \sqrt{\frac{4\omega_0^2 L^2}{R^2} - 1}$$

$\because Q (=100)$ is large

$$= \frac{\omega_0 L}{R} \quad (1)$$

Sol: $\frac{1}{n} \frac{V^2}{R} = \left(\frac{V}{Z}\right)^2 R$

$$\Rightarrow Z^2 = nR^2$$

$\because P_{\max}$ is generated at resonance

$$\Rightarrow R^2 + (X_L - X_C)^2 = nR^2$$

Let new ω be $t\omega_0$.

$$\Rightarrow (n-1) = \frac{1}{R^2} \left[(t\omega_0 L)^2 - \frac{1}{(t\omega_0 C)^2} \right]^2$$

$$\Rightarrow n-1 = \frac{L^2 \omega_0^2}{R^2} \left[t - \frac{1}{t\omega_0^2 LC} \right]^2$$

$$(1) \Rightarrow n-1 = Q^2 \left(t - \frac{1}{t} \right)^2$$

Lets say frequency is increased. So $t > 1/t$

$$\Rightarrow \sqrt{n-1} = Q \left(t - \frac{1}{t} \right) \quad \left\{ \begin{array}{l} n \rightarrow 2 \\ Q = 100 \end{array} \right\}$$

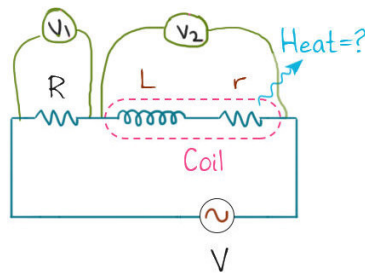
$$\Rightarrow 1 = 100 \left(t - \frac{1}{t} \right)$$

$$\Rightarrow t = \frac{201}{200}$$

$$\therefore \% \text{ change in } \omega = \frac{t\omega_0 - \omega_0}{\omega_0} \times 100 = (t-1)100 = 0.5\%$$

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4.141. A series circuit consisting of an inductance-free resistance $R = 0.16 \text{ k}\Omega$ and a coil with active resistance is connected to the mains with effective voltage $V = 220 \text{ V}$. Find the heat power generated in the coil if the effective voltage values across the resistance R and the coil are equal to $V_1 = 80 \text{ V}$ and $V_2 = 180 \text{ V}$ respectively.



$$\text{Heat in coil: } = i^2 r = \left(\frac{V_1}{R} \right)^2 r$$

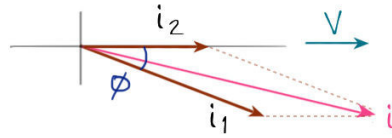
$$V_1 = \frac{VR}{Z} = \frac{VR}{\sqrt{(R+r)^2 + X_L^2}} \quad (1)$$

$$V_2 = V \frac{\sqrt{r^2 + X_L^2}}{Z} = \frac{V \sqrt{r^2 + X_L^2}}{\sqrt{(R+r)^2 + X_L^2}} \quad (2)$$

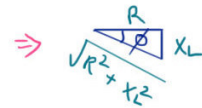
$$\text{From 1,2: } r = \frac{1}{2} \left(\frac{V^2 - V_1^2 - V_2^2}{V_1^2} \right) R$$

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4.142. A coil and an inductance-free resistance $R = 25 \Omega$ are connected in parallel to the ac mains. Find the heat power generated in the coil provided a current $I = 0.90 \text{ A}$ is drawn from the mains. The coil and the resistance R carry currents $I_1 = 0.50 \text{ A}$ and $I_2 = 0.60 \text{ A}$ respectively.



$$\tan \phi = \frac{X_L}{R}$$



$$\text{Heat in coil} = i_1^2 r$$

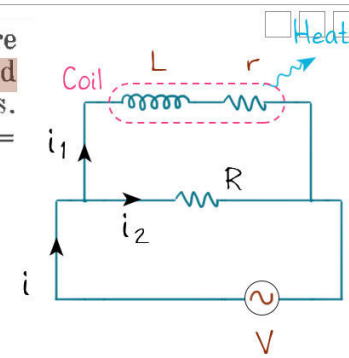
$$1 \quad V = i_2 R$$

$$2 \quad V = i_1 \sqrt{r^2 + X_L^2}$$

$$3 \quad i^2 = i_1^2 + i_2^2 + 2i_1 i_2 \cos \phi$$

$$\frac{R}{\sqrt{R^2 + X_L^2}}$$

$$\text{From 1, 2, 3: } r = \frac{R}{2} \left(\frac{i^2 - i_1^2 - i_2^2}{i_1^2} \right)$$

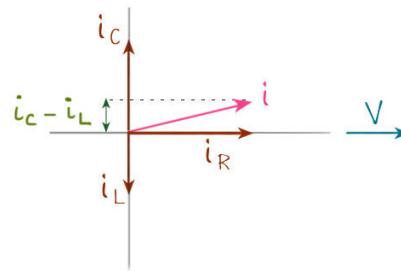
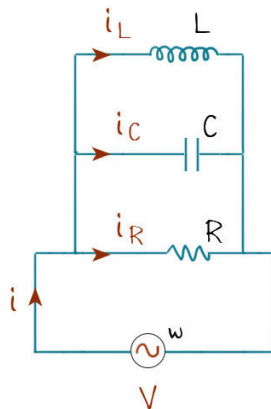


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4.143. An alternating current of frequency $\omega = 314 \text{ s}^{-1}$ is fed to a circuit consisting of a capacitor of capacitance $C = 73 \mu\text{F}$ and an active resistance $R = 100 \Omega$ connected in parallel. Find the impedance of the circuit.

Lets take a general case of L, C, R in parallel.

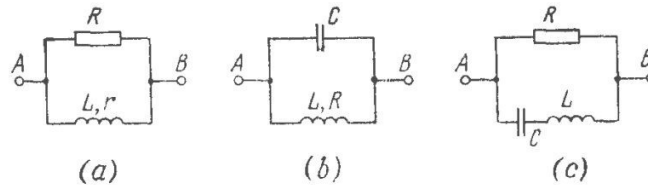
$$\text{Impedance } Z = \frac{V}{i}$$



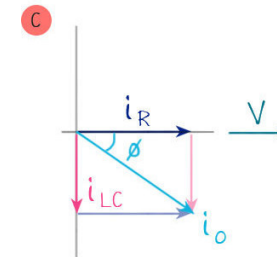
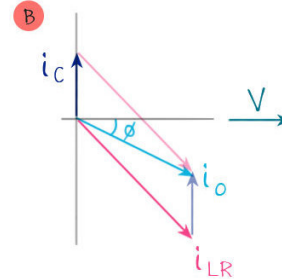
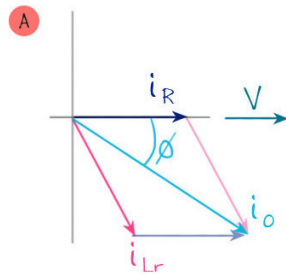
$$\begin{aligned} &= \frac{V}{\sqrt{i_R^2 + (i_C - i_L)^2}} \\ &= \frac{V}{\sqrt{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_C} - \frac{V}{X_L}\right)^2}} \\ &= \frac{1}{\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}} \end{aligned}$$

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4.144. Draw the approximate vector diagrams of currents in the circuits shown in Fig. 4.35. The voltage applied across the points A and B is assumed to be sinusoidal; the parameters of each circuit are so chosen that the total current I_0 lags in phase behind the external voltage by an angle ϕ .

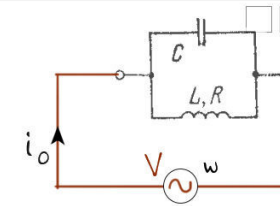


∴ Net i lags behind V , effect of L is more than C .

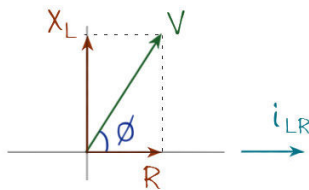


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4.146. A capacitor with capacitance C and a coil with active resistance R and inductance L are connected in parallel to a source of sinusoidal voltage of frequency ω . Find the phase difference between the current fed to the circuit and the source voltage.



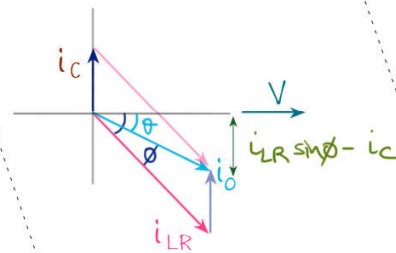
In LR branch:



$$i_{LR} = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\tan \phi = \frac{X_L}{R} = \frac{\omega L}{R} \Rightarrow \text{triangle with sides } R, \omega L, \sqrt{R^2 + \omega^2 L^2}$$

In complete circuit:



$$i_c = \frac{V}{X_C} = V\omega C$$

∴ Current i_0 lags V by θ

$$\tan \theta = \frac{i_{LR} \sin \phi - i_c}{i_{LR} \cos \phi}$$

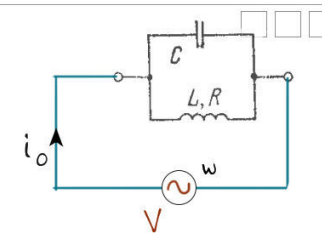
$$= \tan \phi - \frac{i_c}{i_{LR} \cos \phi}$$

$$= \frac{\omega L}{R} - \frac{\omega C}{\frac{R}{\sqrt{R^2 + \omega^2 L^2}}}$$

$$= \frac{\omega L}{R} - \frac{\omega C (R^2 + \omega^2 L^2)}{R}$$

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4.147. A circuit consists of a capacitor with capacitance C and a coil with active resistance R and inductance L connected in parallel. Find the impedance of the circuit at frequency ω of alternating voltage.



Impedance: $Z = \frac{V}{I_0}$

From 4.146: $I_0 = \sqrt{I_C^2 + I_{LR}^2 + 2I_C I_{LR} \cos(90^\circ + \phi)}$

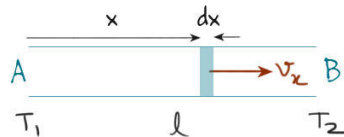
$$= \sqrt{(V\omega C)^2 + \left(\frac{V}{\sqrt{R^2 + (\omega L)^2}}\right)^2 + 2(V\omega C)\left(\frac{V}{\sqrt{R^2 + (\omega L)^2}}\right)\left(\frac{-\omega L}{\sqrt{R^2 + (\omega L)^2}}\right)}$$

$$= V \sqrt{\omega^2 C^2 + \frac{1}{R^2 + \omega^2 L^2} - \frac{2\omega^2 LC}{R^2 + \omega^2 L^2}}$$

$$\therefore Z = \frac{V}{I_0} = \frac{1}{\sqrt{\omega^2 C^2 + \frac{1 - 2\omega^2 LC}{R^2 + \omega^2 L^2}}} = \sqrt{\frac{R^2 + \omega^2 L^2}{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}} //$$

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4.150. How long will it take sound waves to travel the distance l between the points A and B if the air temperature between them varies linearly from T_1 to T_2 ? The velocity of sound propagation in air is equal to $v = \alpha\sqrt{T}$, where α is a constant.



$$v = \alpha\sqrt{T}$$

$$\Rightarrow v_x = \alpha\sqrt{T_x}$$

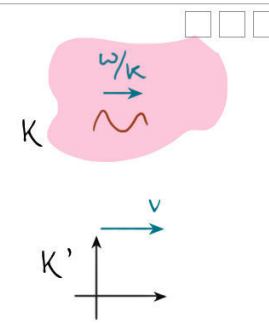
$$\Rightarrow \frac{dx}{dt} = \alpha\sqrt{T_1 + (T_2 - T_1)\frac{x}{l}} \quad (\because T \text{ varies linearly})$$

$$\Rightarrow \int_0^l \frac{dx}{\sqrt{T_1 + (T_2 - T_1)\frac{x}{l}}} = \alpha \int_0^t dt$$

$$\Rightarrow t = \frac{2l}{\alpha(\sqrt{T_1} + \sqrt{T_2})} //$$

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4.153. A plane elastic wave $\xi = a \cos(\omega t - kx)$ propagates in a medium K . Find the equation of this wave in a reference frame K' moving in the positive direction of the x axis with a constant velocity V relative to the medium K . Investigate the expression obtained.



Wave velocity in K frame = $\frac{\omega}{k}$ $\rightarrow$ Wave equation: $y = a \cos k \left(\frac{\omega}{k} t - x \right)$

Wave velocity in K' frame = $\frac{\omega}{k} - V$ $\rightarrow$ $\therefore$ Wave equation: $y = a \cos k \left[\left(\frac{\omega}{k} - V \right) t - x \right]$ //

Note: w.r.t moving frame, wave velocity changes, but wave number ($k = \frac{1}{\lambda}$) and amplitude (a) remain same.

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4.154. Demonstrate that any differentiable function $f(t + \alpha x)$, where α is a constant, provides a solution of wave equation. What is the physical meaning of the constant α ?

We know, wave equation in differential form is: $\frac{\partial^2 e}{\partial t^2} = v^2 \frac{\partial^2 e}{\partial x^2}$ ①

Sol:

$$e = f(t + \alpha x) \quad \text{②}$$

$$\frac{\partial^2 e}{\partial t^2} = f''(t + \alpha x)$$

$$\frac{\partial^2 e}{\partial x^2} = \alpha^2 f''(t + \alpha x)$$

$$\frac{\partial^2 e}{\partial t^2} = \frac{1}{\alpha^2} \frac{\partial^2 e}{\partial x^2} \quad \text{③}$$

③ is of same form as ①

$\therefore$ ② indeed represents a wave.

and constant α is inverse of wave velocity. i.e: $\alpha = \pm \frac{1}{v}$

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4.155. The equation of a travelling plane sound wave has the form $\xi = 60 \cos (1800t - 5.3x)$, where ξ is expressed in micrometres, t in seconds, and x in metres. Find:

- the ratio of the displacement amplitude, with which the particles of medium oscillate, to the wavelength;
- the velocity oscillation amplitude of particles of the medium and its ratio to the wave propagation velocity;
- the oscillation amplitude of relative deformation of the medium and its relation to the velocity oscillation amplitude of particles of the medium.

$$\textcircled{A} \frac{A}{\lambda} = \frac{60 \times 10^{-6}}{\left(\frac{2\pi}{5.3}\right)} = 5.1 \times 10^{-5}$$

$\downarrow \frac{2\pi}{k}$

∴ Amplitude of a sound wave is quite small relative to its wavelength.

Our typical wave graph is on a heavily distorted scale.



For 'feeling': If wavelength is our height (1.5m, i.e. 200Hz), amplitude will be of size of our hair thickness (100 micron).



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- the velocity oscillation amplitude of particles of the medium and its ratio to the wave propagation velocity;

$$\xi = 60 \cos (1800t - 5.3x)$$

$$v_{\text{max particle}} = \omega A = 1800 (60 \times 10^{-6}) = 10.8 \text{ cm/s}$$

$$v_{\text{wave}} = \omega / k = 1800 / 5.3 = 340 \text{ m/s}$$

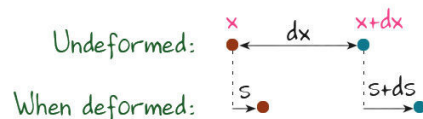
$$\therefore \frac{v_{\text{max particle}}}{v_{\text{wave}}} = \frac{10.8 \text{ cm/s}}{340 \text{ m/s}} = 3.2 \times 10^{-4}$$

Again the velocity of particle is miniscule compared to velocity of wave.

(c) the oscillation **amplitude of relative deformation** of the medium and its relation to the **velocity oscillation amplitude** of particles of the medium. $S = a \cos(\omega t - kx)$

Lets consider two nearby particles at location x and $x+dx$ when particles are at rest.

When wave passes through them, let their displacements be s and $s+ds$.

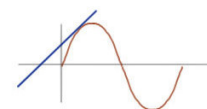


Amplitude of relative deformation

Relative deformation is defined as deformation per unit length (kind of like strain!)

$$\therefore \text{Relative deformation} = \frac{S_{x+dx} - S_x}{dx} = \frac{(s+ds) - (s)}{dx} = \frac{\partial S}{\partial x} = a k \sin(\omega t - kx) \quad (1)$$

= Slope of $s-x$ graph



Also, velocity: $= \frac{\partial S}{\partial t} = -a\omega \sin(\omega t - kx) \quad (2)$

Amplitude of velocity

$$\frac{(2)}{(1)} \Rightarrow \frac{\partial S}{\partial t} = -\frac{\omega}{k} \frac{\partial S}{\partial x} \Rightarrow \frac{\partial S}{\partial t} = -v \frac{\partial S}{\partial x}$$

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4.156. A plane wave $\xi = a \cos(\omega t - kx)$ propagates in a homogeneous elastic medium. For the moment $t = 0$ draw

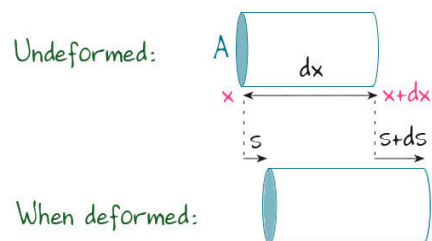
(a) the plots of ξ , $\partial \xi / \partial t$, and $\partial \xi / \partial x$ vs x ;

(b) the **velocity direction of the particles of the medium** at the points where $\xi = 0$, for the cases of **longitudinal and transverse waves**;

(c) the approximate plot of density distribution $\rho(x)$ of the medium for the case of **longitudinal waves**.

Finding density function from $e = a \cos(\omega t - kx)$

Just like 4.155(c), lets take two nearby points.



(c) mass = constant

$$\Rightarrow \rho V = \text{const}$$

$$\Rightarrow \rho dV + V d\rho = 0$$

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V} = -\frac{(A ds)}{A dx}$$

$$\Rightarrow d\rho = -\rho \frac{ds}{dx} \quad (1)$$

For given equation:

$$\frac{ds}{dx} = a k \sin(\omega t - kx) \quad (2)$$

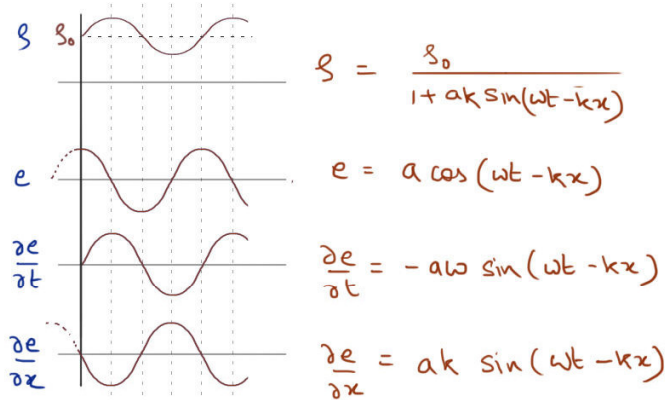
Also: $\rho = \rho_0 + d\rho$

$$\Rightarrow \rho = \frac{\rho_0}{1 + \frac{ds}{dx}}$$

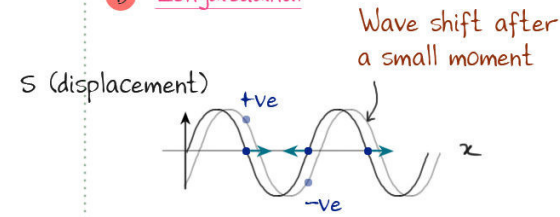
$$\Rightarrow \rho = \frac{\rho_0}{1 + a k \sin(\omega t - kx)}$$

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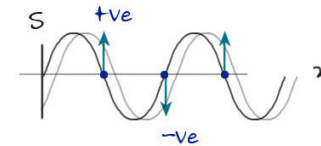
A Plots



B Longitudinal



Transverse



(b) the velocity direction of the particles of the medium at the points where $\xi = 0$, for the cases of longitudinal and transverse waves;

Let the direction of wave be towards right.
So representing particle velocity direction with arrows:

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4.157. A plane elastic wave $\xi = ae^{-\gamma x} \cos(\omega t - kx)$, where a , γ , ω , and k are constants, propagates in a homogeneous medium. Find the phase difference between the oscillations at the points where the particles' displacement amplitudes differ by $\eta = 1.0\%$, if $\gamma = 0.42 \text{ m}^{-1}$ and the wavelength is $\lambda = 50 \text{ cm}$.

We know:

$$\frac{\Delta x}{\lambda} = \frac{\Delta \phi}{2\pi} \quad (1)$$

Given equation is for a damped wave. As explained in book:

• Equations of plane and spherical waves:

$$\xi = a \cos(\omega t - kx), \quad \xi = \frac{C}{r} \cos(\omega t - kr). \quad (4.3a)$$

a constant

In the case of a homogeneous absorbing medium the factors $e^{-\gamma x}$ and $e^{-\gamma r}$ respectively appear in the formulas, where γ is the wave damping coefficient.

Sol: Amplitude = $a e^{-\gamma x}$

Given: $\frac{a e^{-\gamma(x+\Delta x)} - a e^{-\gamma x}}{a e^{-\gamma x}} = \eta$

$\Rightarrow \frac{e^{-\gamma \Delta x} - 1}{1} = \eta$

$\Delta x = \frac{-\ln(1+\eta)}{\gamma} \quad (2)$

From 1,2:

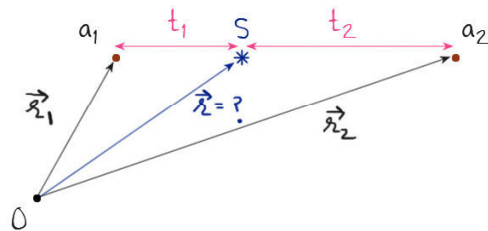
$\approx \eta$ ($\because \eta$ is small)

$$|\Delta \phi| = \frac{2\pi}{\lambda} \ln(1+\eta) = \frac{2\pi\eta}{\lambda}$$

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4.158. Find the radius vector defining the position of a point source of spherical waves if that source is known to be located on the straight line between the points with radius vectors $\vec{r}_1$ and $\vec{r}_2$ at which the oscillation amplitudes of particles of the medium are equal to a_1 and a_2 . The damping of the wave is negligible, the medium is homogeneous.

Let distances be t_1 and t_2 .



For the spherical wavefronts reaching radius t_1 and t_2 :

$$P_1 = P_2$$

$$\therefore \text{From 1: } k a_1^2 t_1^2 = k a_2^2 t_2^2$$

$$\Rightarrow \frac{t_1}{t_2} = \frac{a_2}{a_1} \quad (3)$$

We know from vectors:

$$\vec{r} = \frac{t_2 \vec{r}_1 + t_1 \vec{r}_2}{t_1 + t_2} \quad (2)$$

$$\text{From 2, 3: } \vec{r} = \frac{a_1 \vec{r}_1 + a_2 \vec{r}_2}{a_1 + a_2}$$

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We know:

$$P = \frac{1}{2} \rho v \omega^2 s_0^2 A$$

$$4\pi r^2$$

For point source:

$$\Rightarrow P = k s_0^2 \lambda^2 \quad (1)$$

constant

4.159. A point isotropic source generates sound oscillations with frequency $\nu = 1.45$ kHz. At a distance $r_0 = 5.0$ m from the source the displacement amplitude of particles of the medium is equal to $a_0 = 50$ μm , and at the point A located at a distance $r = 10.0$ m from the source the displacement amplitude is $\eta = 3.0$ times less than a_0 . Find:

(a) the damping coefficient γ of the wave;

(b) the velocity oscillation amplitude of particles of the medium at the point A.

Due to point source in ideal medium: $a_1 r_1 = a_2 r_2$ (refer 4.158)

But here: $a_0 r_0 > a_1 r_1$ $\therefore$ the medium damps the wave.

$$\begin{matrix} 50 & 5 & 50 & 10 \\ & & 3 & \end{matrix}$$

Let damped wave equation for particle displacement be: $S = \frac{C}{r} e^{-\gamma r} \cos(\omega t - \kappa r) \quad (1)$ (refer 4.157)

$$\text{A Given: } \frac{e^{-\gamma r_0}}{r_0} = \eta \frac{e^{-\gamma r_1}}{r_1}$$

$$\Rightarrow \gamma = \frac{1}{r_2 - r_0} \ln\left(\frac{\eta r_0}{r_1}\right) //$$

B Differentiating 1:

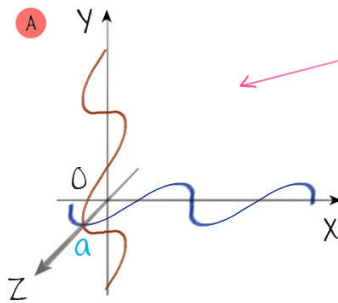
$$v_r = \frac{\partial S}{\partial t} \Big|_{r=r_1} = \frac{C}{r_1} e^{-\gamma r_1} \omega (-) \sin(\omega t - \kappa r_1)$$

$$\text{velocity amplitude} = \frac{a_0 \omega}{\eta} = \frac{a_0 (2\pi \nu)}{\eta}$$

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4.160. Two plane waves propagate in a homogeneous elastic medium, one along the x axis and the other along the y axis: $\xi_1 = a \cos(\omega t - kx)$, $\xi_2 = a \cos(\omega t - ky)$. Find the wave motion pattern of particles in the plane xy if both waves

- are transverse and their oscillation directions coincide;
- are longitudinal.



As their displacement directions coincide, that direction must be along Z axis.

We have shown in graph, displacements due to just two waves that pass through origin at $t=0$. We can observe constructive interference at O .

Given:

$$Z_1 = a \cos(\omega t - kx)$$

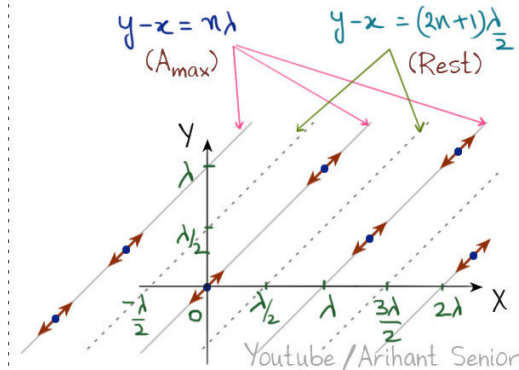
$$Z_2 = a \cos(\omega t - ky)$$

$$\therefore Z = Z_1 + Z_2 = 2a \cos \frac{k}{2}(y-x) \cos \left[\omega t - \frac{k(x+y)}{2} \right]$$

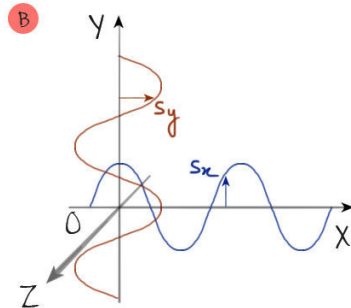
$$\therefore \text{Net Amplitude} = 2a \cos \frac{\pi(y-x)}{\lambda}$$

$\frac{\pi}{\lambda} (A_{\max})$
 $(2n+1) \frac{\pi}{2} (\text{Rest})$

Z-Motion of particles on planes is shown



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As waves are longitudinal, displacements due to each wave will be along the direction of wave.

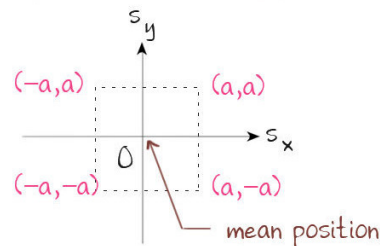
$$\therefore s_x = a \cos(\omega t - kx) \hat{i}$$

$$s_y = a \cos(\omega t - ky) \hat{j}$$

We have shown in graph, displacements due to just two waves that pass through origin at $t=0$.

We can observe constructive interference at O , with net displacement being $a\hat{i} + a\hat{j}$

Displacement of every particle will be within the shown square.



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Displacements are: $S_x = a \cos(\omega t - kx) \hat{i}$ and $S_y = a \cos(\omega t - ky) \hat{j}$

At any given time t ,

$$\Delta\phi = (\omega t - kx) - (\omega t - ky) = k(y-x) = \frac{2\pi(y-x)}{\lambda}$$

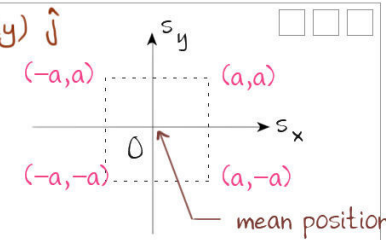
Let phase of s_x be θ

A $2n\pi$ $S_x = a \cos \theta$
 $S_y = a \cos \theta$ $\Rightarrow S_x = S_y$

B $(2n+1)\pi$ $S_x = a \cos \theta$
 $S_y = -a \cos \theta$ $\Rightarrow S_x = -S_y$

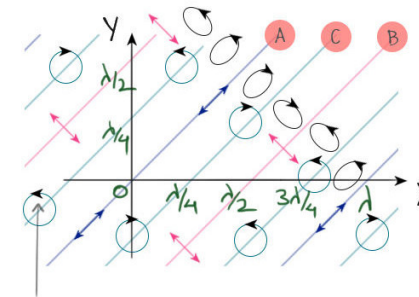
C $(2n+1)\frac{\pi}{2}$ $S_x = a \cos \theta$
 $S_y = a \sin \theta$ $\Rightarrow S_x^2 + S_y^2 = a^2$

D other $S_x = a \cos \theta$
 $S_y = a \cos(\theta - \Delta\phi)$ $\xrightarrow{\text{eliminate } \theta}$ equation of ellipse



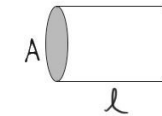
XY-Motion of particles on planes is shown

- A $y-x = n\lambda$ C $y-x = (2n+1)\frac{\lambda}{4}$
 B $y-x = (2n+1)\frac{\lambda}{2}$ D ellipses in remaining



arrows show sense of rotation

4.161. A plane undamped harmonic wave propagates in a medium. Find the mean space density of the total oscillation energy $\langle W \rangle$, if at any point of the medium the space density of energy becomes equal to W_0 , one-sixth of an oscillation period after passing the displacement maximum.



$$T/6 \Rightarrow \Delta\phi = \frac{2\pi}{6}$$

For: $S = S_0 \sin(\omega t - kx)$

$$P = \rho v \omega^2 S_0^2 A \cos^2(\omega t - kx)$$

$$E_{\text{density}}: W = \frac{E}{V} = \frac{P \cdot t}{Al} = \frac{\rho v \omega^2 S_0^2 A \cos^2(\omega t - kx) \cdot \frac{l}{v}}{Al} = \rho \omega^2 S_0^2 \cos^2(\omega t - kx)$$

Sol: $S = S_0 \sin(\omega t - kx)$

At maximum displacement, phase = $\pi/2$

At $T/6$ after that, phase = $\pi/2 + \frac{2\pi}{6}$

$$W_0 = \rho \omega^2 S_0^2 \cos^2\left(\frac{\pi}{2} + \frac{2\pi}{6}\right)$$

$$\Rightarrow W_0 = \rho \omega^2 S_0^2 \cdot \frac{3}{4} \quad (1)$$

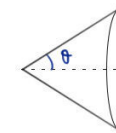
$$\langle W \rangle = \frac{\rho \omega^2 S_0^2}{2} \quad (2)$$

From 1, 2:

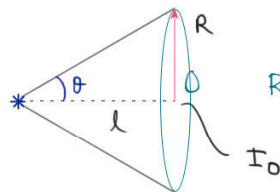
$$\langle W \rangle = \frac{1}{2} \cdot \frac{4}{3} W_0 = \frac{2W_0}{3} //$$

Youtube / Arihant Senior

4.162. A point isotropic sound source is located on the perpendicular to the plane of a ring drawn through the centre O of the ring. The distance between the point O and the source is $l = 1.00$ m, the radius of the ring is $R = 0.50$ m. Find the mean energy flow across the area enclosed by the ring if at the point O the intensity of sound is equal to $I_0 = 30 \mu\text{W/m}^2$. The damping of the waves is negligible.



$$\text{Solid angle} = 2\pi(1 - \cos\theta)$$



$$\text{Required energy flow through ring} = \frac{\text{Solid angle subtended by ring}}{\text{Total solid angle}} \times \text{Total energy} \quad \text{IA}$$

$$= \frac{2\pi(1 - \cos\theta)}{4\pi} \times [I_0 \cdot 4\pi R^2]$$

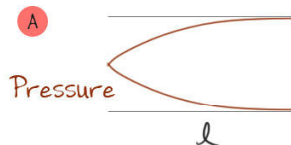
$$= \frac{1 - \frac{l}{\sqrt{l^2 + R^2}}}{2} I_0 (4\pi R^2)$$

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4.170. Find the number of possible natural oscillations of air column in a pipe whose frequencies lie below $\nu_0 = 1250$ Hz. The length of the pipe is $l = 85$ cm. The velocity of sound is $v = 340$ m/s. Consider the two cases:

- the pipe is closed from one end;
- the pipe is opened from both ends.

The open ends of the pipe are assumed to be the antinodes of displacement.



$$l = \left(n + \frac{1}{2}\right) \frac{\lambda}{2} \quad \text{G.I.F}$$

$$\Rightarrow n = \left[\frac{2fl}{v} - \frac{1}{2} \right]$$

$$= [5.75] = 5$$

$\therefore$ Possible oscillations = 0, 1, 2, 3, 4, 5 $\rightarrow 6$



$$l = \frac{n\lambda}{2}$$

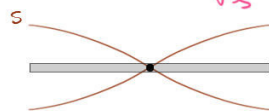
$$\Rightarrow n = \left[\frac{2fl}{v} \right]$$

$$= [6.75] = 6$$

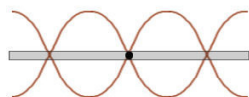
$\therefore$ Possible oscillations = 1, 2, 3, 4, 5, 6 $\rightarrow 6$

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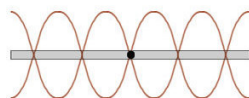
4.171. A copper rod of length $l = 50$ cm is clamped at its midpoint. Find the number of natural longitudinal oscillations of the rod in the frequency range from 20 to 50 kHz. What are those frequencies equal to? $v_{Cu} = \sqrt{\frac{Y}{\rho}} = 4600$ m/s



$$l = \frac{\lambda}{2}$$



$$l = \lambda + \frac{\lambda}{2}$$



$$l = 2\lambda + \frac{\lambda}{2}$$

∴ For natural oscillations: $l = n\lambda + \frac{\lambda}{2}$
 $n = 0, 1, 2, \dots$

$$\Rightarrow n = \left[\frac{fl}{v} - \frac{1}{2} \right] \text{ G.I.F.}$$

$$n_{\min} = \left[\frac{20000 (0.5)}{4600} - \frac{1}{2} \right] = [1.67] = 1$$

$$n_{\max} = \left[\frac{50000 (0.5)}{4600} - \frac{1}{2} \right] = [4.9] = 4$$

∴ Range of n : 1, 2, 3, 4

and possible oscillations for $n = 1, 2, 3, 4 \rightarrow 4$

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4.172. A string of mass m is fixed at both ends. The fundamental tone oscillations are excited with circular frequency ω and maximum displacement amplitude $a_{\max}$. Find:

- the maximum kinetic energy of the string;
- the mean kinetic energy of the string averaged over one oscillation period.

Let wave equation be: $y = a \sin kx \cos \omega t$

$$\Rightarrow v_x = \frac{\partial y}{\partial t} = -\omega a \sin kx \sin \omega t$$

KE of elemental string is:

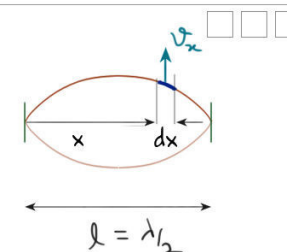
$$dKE = \frac{1}{2} dm v_x^2$$

$$= \frac{1}{2} \frac{dx}{l} m (-\omega a \sin kx \sin \omega t)^2$$

$$\Rightarrow KE = \frac{m\omega^2 a^2 \sin^2 \omega t}{2l} \int_0^l \sin^2 kx dx$$

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2l} = \frac{\pi}{l}$

$$= \frac{m\omega^2 a^2}{4} \sin^2 \omega t \quad \text{①}$$



From 1:

$$\text{A } KE_{\max} = \frac{m\omega^2 a^2}{4}$$

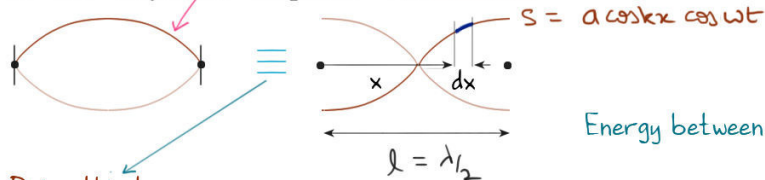
$$\text{B } \langle KE \rangle = \frac{m\omega^2 a^2}{4} \frac{\int_0^T \sin^2 \omega t dt}{T}$$

$\int_0^T \sin^2 \omega t dt = \frac{T}{2}$

$$= \frac{m\omega^2 a^2}{8}$$

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4.173. A standing wave $\xi = a \sin kx \cdot \cos \omega t$ is maintained in a homogeneous rod with cross-sectional area S and density ρ . Find the total mechanical energy confined between the sections corresponding to the adjacent displacement nodes.



Energy between nodes = Energy between antinodes

Doing this because:

We have already derived following energy densities in 4.165 for equation $s = a \cos kx \cos \omega t$

$$\frac{dPE}{dV} = \frac{1}{2} \rho a^2 \omega^2 \sin^2 kx \cos^2 \omega t \quad \frac{dKE}{dV} = \frac{1}{2} \rho a^2 \omega^2 \cos^2 kx \sin^2 \omega t$$

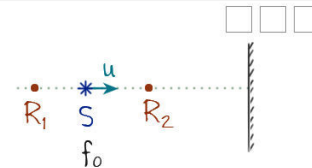
Putting: $dV = S dx$, we can write:

$$E = \int dPE + \int dKE = \frac{1}{2} \rho S a^2 \omega^2 \left[\cos^2 \omega t \int_0^{\lambda/2} \sin^2 kx dx + \sin^2 \omega t \int_0^{\lambda/2} \cos^2 kx dx \right]$$

$$= \frac{1}{2} \rho S a^2 \omega^2 \frac{\lambda}{4} = \frac{\pi \rho S a^2 \omega^2}{4k} //$$

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4.174. A source of sonic oscillations with frequency $f_0 = 1000$ Hz moves at right angles to the wall with a velocity $u = 0.17$ m/s. Two stationary receivers R_1 and R_2 are located on a straight line, coinciding with the trajectory of the source, in the following succession: R_1 -source- R_2 -wall. Which receiver registers the beatings and what is the beat frequency? The velocity of sound is equal to $v = 340$ m/s.



Wall will reflect the same frequency it receives.

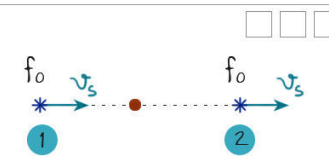
∴ Frequency received by wall = $\left(\frac{v}{v-u}\right) f_0$ = Frequency reflected from wall:

At R_1 From source S: $\left(\frac{v}{v+u}\right) f_0$ From wall: $\left(\frac{v}{v-u}\right) f_0$ beats $\rightarrow f_0 v \left(\frac{1}{v-u} - \frac{1}{v+u} \right) = f_0 \frac{2uv}{v^2 - u^2} //$

At R_2 From source S: $\left(\frac{v}{v-u}\right) f_0$ From wall: $\left(\frac{v}{v-u}\right) f_0$ No beats

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4.175. A stationary observer receives sonic oscillations from two tuning forks one of which approaches, and the other recedes with the same velocity. As this takes place, the observer hears the beatings with frequency $f = 2.0$ Hz. Find the velocity of each tuning fork if their oscillation frequency is $f_0 = 680$ Hz and the velocity of sound in air is $v = 340$ m/s.



$$\begin{aligned} \text{Beats} = f &= f_1 - f_2 \\ &= f_0 \left(\frac{v}{v - v_s} - \frac{v}{v + v_s} \right) \end{aligned}$$

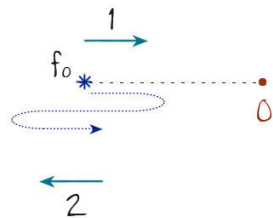
$$\Rightarrow f = f_0 \frac{2v_s v}{v^2 - v_s^2}$$

↓ Solving quadratic in v_s and rejecting negative solution:

$$v_s = v \frac{f_0}{f} \left(\sqrt{1 + \left(\frac{f}{f_0}\right)^2} - 1 \right) \approx \frac{1}{2} v \frac{f_0}{f} //$$

Youtube / Arihant Senior

4.176. A receiver and a source of sonic oscillations of frequency $f_0 = 2000$ Hz are located on the x axis. The source swings harmonically along that axis with a circular frequency ω and an amplitude $a = 50$ cm. At what value of ω will the frequency bandwidth registered by the stationary receiver be equal to $\Delta f = 200$ Hz? The velocity of sound is equal to $v = 340$ m/s.



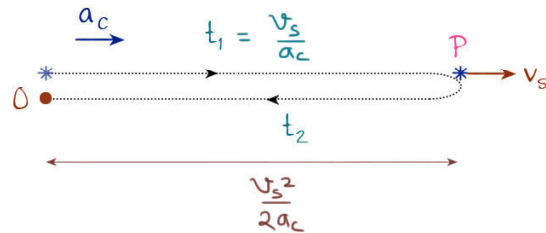
$$\begin{aligned} \Delta f &= f_1 - f_2 \\ &= f_0 \left(\frac{v}{v - v_s} - \frac{v}{v + v_s} \right) \\ &= f_0 \frac{2v_s v}{v^2 - v_s^2} \quad \text{(same as in 4.175)} \\ &\quad \therefore \text{Same solution} \end{aligned}$$

$$\therefore v_s = \omega a = v \frac{f_0}{\Delta f} \left(\sqrt{1 + \left(\frac{\Delta f}{f_0}\right)^2} - 1 \right) //$$

Cant approximate here like we did in 4.175 as $\frac{\Delta f}{f_0}$ is quite high.
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4.177. A source of sonic oscillations with frequency $\nu_0 = 1700$ Hz and a receiver are located at the same point. At the moment $t = 0$ the source starts receding from the receiver with constant acceleration $a_c = 10.0$ m/s². Assuming the velocity of sound to be equal to $v = 340$ m/s, find the oscillation frequency registered by the stationary receiver $t = 10.0$ s after the start of motion.

Lets say sound pulse emitted by source at point P is the one that reaches O in given time $t = 10$ s.
And velocity of source at P be v_s



t_1 : time taken by Source to reach P

t_2 : time taken by sound (emitted at P) to reach Observer.

$$t = t_1 + t_2$$

$$\Rightarrow t = \frac{v_s}{a_c} + \frac{v_s^2}{2a_c \cdot v}$$

$$\Rightarrow v_s = -v + \sqrt{v^2 + 2a_c t v}$$

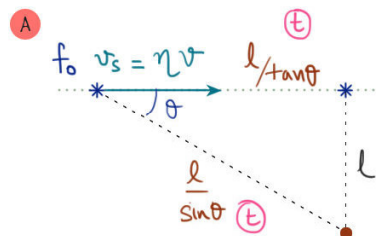
$$\therefore f = f_0 \left(\frac{v}{v + v_s} \right) = \frac{f_0}{\sqrt{1 + \frac{2a_c t v}{v^2}}}$$

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4.178. A source of sound with natural frequency $f_0 = 1.8$ kHz moves uniformly along a straight line separated from a stationary observer by a distance $l = 250$ m. The velocity of the source is equal to $\eta = 0.80$ fraction of the velocity of sound. Find:

(a) the frequency of sound received by the observer at the moment when the source gets closest to him;

(b) the distance between the source and the observer at the moment when the observer receives a frequency $f = f_0$.



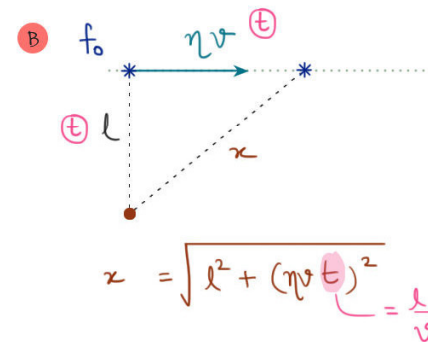
$$\therefore f = \frac{v f_0}{v - v_s \cos \theta}$$

$$= \frac{f_0}{1 - \eta^2}$$

$$t_{\text{source}} = t_{\text{sound}}$$

$$\Rightarrow \frac{l/\tan \theta}{\eta v} = \frac{l/\sin \theta}{v}$$

$$\Rightarrow \eta = \cos \theta$$



$$x = \sqrt{l^2 + (\eta v t)^2} = \frac{l}{\eta}$$

$$= l \sqrt{1 + \eta^2}$$

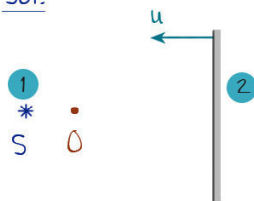
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4.179. A stationary source sends forth monochromatic sound. A wall approaches it with velocity $u = 33 \text{ cm/s}$. The propagation velocity of sound in the medium is $v = 330 \text{ m/s}$. In what way and how much, in per cent, does the wavelength of sound change on reflection from the wall?

For a stationary observer

Concept: A wall reflects same frequency it receives.

Sol:



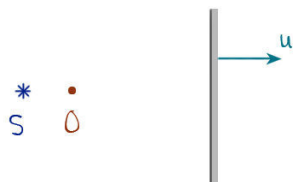
$$f \text{ received by wall: } f' = f_0 \left(\frac{v+u}{v} \right) = f \text{ reflected by wall}$$

$$\therefore f \text{ received by O from wall: } f'' = f' \left(\frac{v}{v-u} \right) = f_0 \left(\frac{v+u}{v-u} \right)$$

$$\text{change in } \lambda = \frac{\frac{v}{f''} - \frac{v}{f_0}}{\frac{v}{f_0}} = \frac{f_0}{f''} - 1 = \left(\frac{v-u}{v+u} \right) - 1 = \frac{-2u}{v+u} = -0.02$$

$\therefore \lambda$ decreases by 2%
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4.180. A source of sonic oscillations with frequency $\nu_0 = 1700 \text{ Hz}$ and a receiver are located on the same normal to a wall. Both the source and the receiver are stationary, and the wall recedes from the source with velocity $u = 6.0 \text{ cm/s}$. Find the beat frequency registered by the receiver. The velocity of sound is equal to $v = 340 \text{ m/s}$.



From previous problem 4.179: $u \longrightarrow -u$

$$\Rightarrow f'' = f_0 \frac{(v-u)}{(v+u)}$$

$$\therefore \text{Beat frequency} = f_0 - f'' = f_0 \left(1 - \frac{v-u}{v+u} \right) = \frac{2uf_0}{v+u}$$

Youtube/Arihant Senior

4.183. At a distance $r_0 = 20.0$ m from a point isotropic source of sound the loudness level $L_{r_0} = 30.0$ dB. Neglecting the damping of the sound wave, find:

- (a) the loudness level at a distance $r = 10.0$ m from the source;
 (b) the distance from the source at which the sound is not heard.

A $L_{r_0} = 10 \log \frac{I_{r_0}}{I_0}$

$L_r = 10 \log \frac{I_r}{I_0}$

Subtracting:

$L_{r_0} - L_r = 10 \log \frac{I_{r_0}}{I_r} = \frac{r^2}{r_0^2}$

$\Rightarrow L_r = L_{r_0} - 20 \log \frac{r}{r_0}$
 $= 30 - 20 \log \left(\frac{1}{2}\right)$
 $= 36 \text{ dB}$

$L = 10 \log \frac{I}{I_0} \text{ dB}$

where, $I_0 = 10^{-12} \text{ W/m}^2$ is human threshold.

Sound is not heard when $I = 10^{-12}$
 i.e. $L = 0$

B From 1, when sound is not heard:

$L_r = L_{r_0} - 20 \log \frac{r}{r_0}$

$\Rightarrow r = r_0 10^{L_{r_0}/20}$
 $= 20 10^{1.5}$
 $= 630 \text{ m}$

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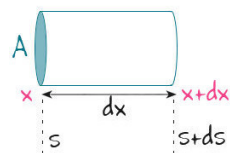
4.185. A plane longitudinal harmonic wave propagates in a medium with density ρ . The velocity of the wave propagation is v . Assuming that the density variations of the medium, induced by the propagating wave, $\Delta \rho \ll \rho$, demonstrate that

(a) the pressure increment in the medium $p = -\rho v^2 (\partial \xi / \partial x)$, where $\partial \xi / \partial x$ is the relative deformation;

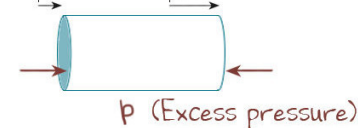
(b) the wave intensity is defined by Eq. (4.3i). $\rightarrow I = \frac{p_0^2}{2 \rho v}$

We know, $B = \frac{p}{-\Delta v/v}$ Excess pressure that causes strain
 $v = \sqrt{\frac{B}{\rho}}$

A Undeformed:



When deformed:



$B = \frac{p}{-\Delta v/v}$

$\Rightarrow \rho v^2 = \frac{p}{-A \Delta s / A dx}$

$\Rightarrow p = -\rho v^2 \frac{\partial s}{\partial x}$

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8 Prove that wave intensity is given by: $I = \frac{p_0^2}{2\rho v}$

Let, $S = S_0 \sin(\omega t - kx) \rightarrow p = BkS_0 \cos(\omega t - kx)$ ①
 $\frac{\partial S}{\partial t} = S_0 \omega \cos(\omega t - kx)$ ②

$F = pA \rightarrow v = \frac{\partial S}{\partial t}$

Transmitted power: $P = Fv$
 $= (pA)v$

∴ Intensity: $I = P/A$
 $= p \cdot v = p \cdot \frac{\partial S}{\partial t}$

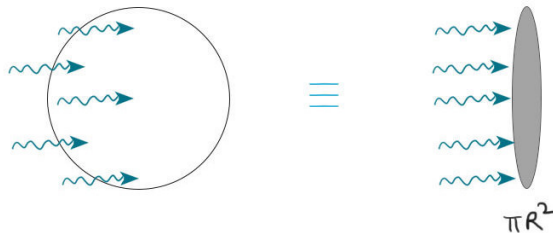
From 1,2:

$I = p_0 S_0 \omega \cos^2(\omega t - kx)$
 $\Rightarrow I = \frac{p_0^2}{B} \left(\frac{\omega}{k} \right) \cos^2(\omega t - kx)$
 $= \frac{p_0^2}{2\rho v} \cos^2(\omega t - kx)$
 $\Rightarrow \langle I \rangle = \frac{p_0^2}{2\rho v} //$

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4.186. A ball of radius $R = 50$ cm is located in the way of propagation of a plane sound wave. The sonic wavelength is $\lambda = 20$ cm, the frequency is $f = 1700$ Hz, the pressure oscillation amplitude in air is $p_0 = 3.5$ Pa. Find the mean energy flow, averaged over an oscillation period, reaching the surface of the ball.

$E_{\text{per second falling on sphere}} = E_{\text{per second falling on circular cross section area}}$



$\langle I \rangle = \frac{p_0^2}{2\rho v}$ (Proven in 4.185)

$\Rightarrow \frac{dE}{dt} = \langle P \rangle = \langle I \rangle \cdot A$

$= \frac{p_0^2}{2\rho v} \cdot \pi R^2$

$= \frac{p_0^2}{2\rho f \lambda} \pi R^2 //$

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4.187. A point A is located at a distance $r = 1.5$ m from a point isotropic source of sound of frequency $f = 600$ Hz. The sonic power of the source is $P = 0.80$ W. Neglecting the damping of the waves and assuming the velocity of sound in air to be equal to $v = 340$ m/s, find at the point A:

(a) the pressure oscillation amplitude p_0 and its ratio to the air pressure;

(b) the oscillation amplitude of particles of the medium; compare it with the wavelength of sound.

$$I = \frac{p_0^2}{2\rho v}$$

$$p_0 = Bk s_0$$

A $I = \frac{P}{A}$

$$\frac{p_0^2}{2\rho v} = \frac{P}{4\pi r^2}$$

$$\Rightarrow p_0 = \sqrt{\frac{2\rho v P}{4\pi r^2}} = 5 \text{ Pa}$$

$$\therefore \frac{p_0}{p_{\text{atm}}} = \frac{5}{10^5} = 5 \times 10^{-5}$$

B $s_0 = \frac{p_0}{Bk}$

$$s_0^2 = \frac{p_0^2}{B^2 k^2} \quad \frac{2\pi}{\lambda} \rightarrow v/f$$

$$= \frac{p_0}{2\pi \rho v f} = 3 \times 10^{-6} \text{ m} = 3 \mu\text{m}$$

$$\therefore \frac{s_0}{\lambda} = \frac{s_0}{v/f} = 5 \times 10^{-6}$$

Youtube / Arihant Senior